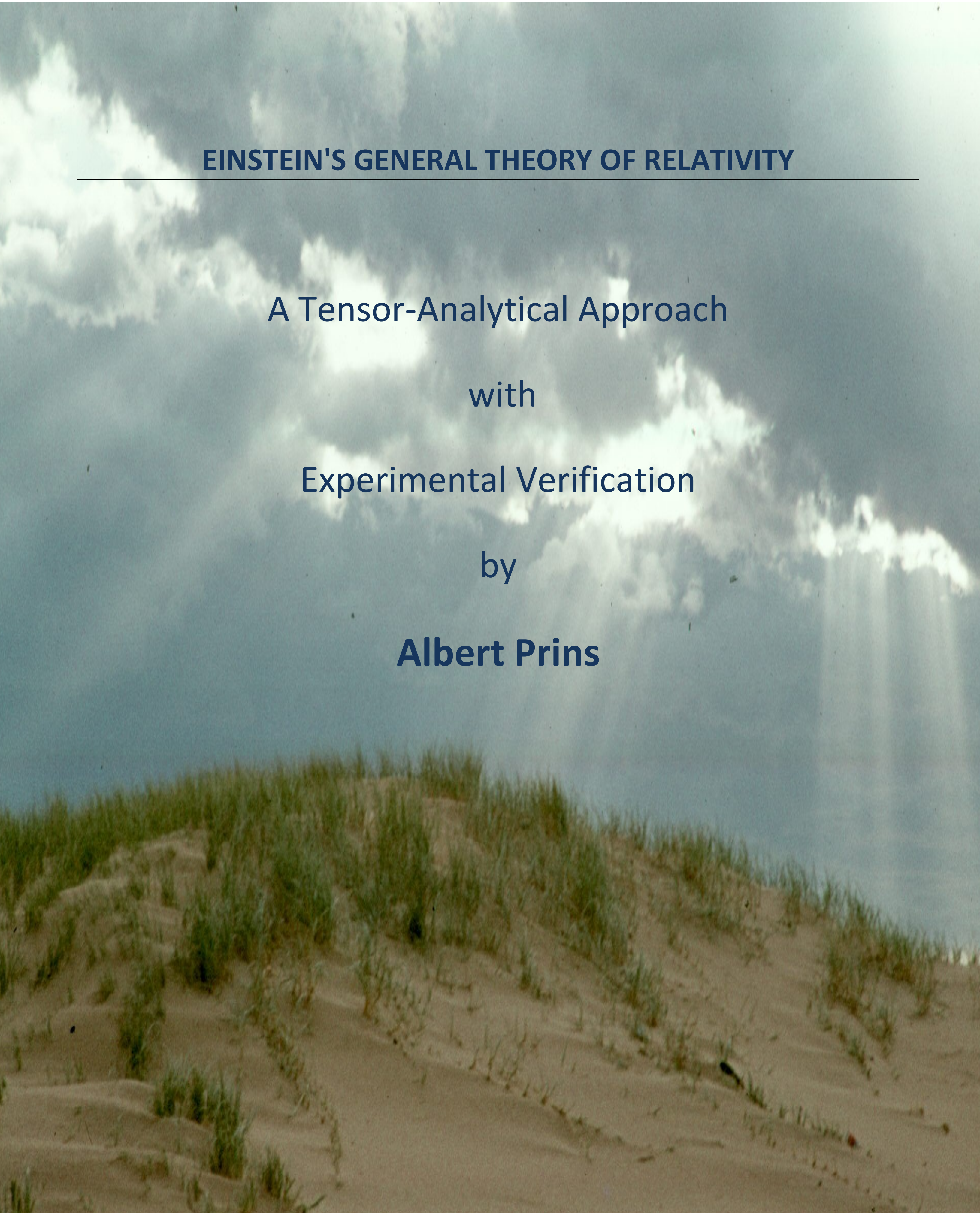

EINSTEIN'S GENERAL THEORY OF RELATIVITY

A Tensor-Analytical Approach

with

Experimental Verification

by

Albert Prins

Foreword

More than a century after its publication, Albert Einstein's general theory of relativity continues to raise questions. From curious readers with a general interest to advanced physics students, many still search for a clear yet mathematically rigorous explanation of this fascinating theory.

This document was written with that goal in mind. We provide a concise overview of the theory itself, along with a systematic derivation of the Einstein field equations, which form the foundation of Einstein's model of space and time.

We also devote substantial attention to experimental evidence supporting the theory, including Mercury's orbital precession, the deflection of light by massive bodies, the Hafele–Keating experiment, and the application of the Schwarzschild solution to a classical projectile trajectory. Each example is carefully developed and explained in both qualitative and quantitative terms.

Our hope is that this work helps to bridge the gap between theory and practice, intuition and formalism, and ultimately between wonder and understanding.

We welcome all forms of feedback. Suggestions, comments, or corrections can be sent to:

aprins@hotmail.com

With kind regards,
Albert Prins

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Table of Contents

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Part I – Introduction and Basic Structure

1 Introduction

The general theory of relativity, formulated by Albert Einstein in 1915, is one of the pillars of modern physics. In this theory, Newton's classical model of gravity is replaced by a geometric approach, in which mass and energy distort the structure of space and time. Objects move in accordance with this distortion.

Gravity is therefore no longer understood as a force that acts at a distance, but as the result of the curvature of space-time. This idea has profound implications for both theoretical and practical physics.

1.1 Purpose of this document

This document aims to provide an in-depth and structured overview of general relativity, with an emphasis on:

- The careful derivation of the underlying mathematical structure, including tensors, covariant derivatives, and field equations.
- The discussion of experiments and applications that support or illustrate the theory.
- Answering frequently asked questions about concepts, formulas, and interpretations within relativistic physics.

It is intended as a bridge between abstract formalism and physical intuition.

1.2 Approach

Unlike popular introductions, this document emphasizes mathematical precision and systematic derivation. We use the following approach:

- We derive step by step the tensor-analytic equations that form the basis of Einstein's field equations.
- We pay extensive attention to coordinate transformations, covariant derivatives, and the use of the metric tensor.
- We apply the Schwarzschild solution to concrete experiments, such as the Hafele–Keating experiment, the deflection of light near the sun, and the precession of Mercury.

We also demonstrate that the Schwarzschild solution satisfies Einstein's field equations, and describe how s**pace**-time curvature is mathematically expressed through the metric and Christoffel symbols.

1.3 Target audience

This text is written for:

- (Geo)physicists and mathematicians who are interested in the mathematical foundations of general relativity.
- Physics students who want to go beyond standard textbooks.
- Interested parties who not only want to know *how* the theory works, but also *why* the equations are structured in this particular way.

A certain basic knowledge of mathematics, particularly linear algebra and differential equations, is recommended.

1.4 Conclusion

With this work, we hope to bridge the gap between theory and practice, between formalism and insight. Each chapter builds on the previous one, but the sections are structured in such a way that they can also be understood separately.

The appendices at the end offer additional explanations, alternative derivations, and applications in related areas such as special relativity, nuclear physics, and rotational mechanics.

Part II – Derivation of General Relativity

2 General Relativity

Before Einstein formulated his famous general theory of relativity in 1915, he first developed the **special theory of relativity** (see Appendix 9), which he published in 1905. This theory described physical phenomena in inertial (non-accelerating) coordinate systems – i.e., systems that move at a constant speed relative to each other. The influence of mass and gravity was not yet taken into account.

The special theory of relativity is based on two fundamental principles:

1. **The speed of light in a vacuum is constant** and is the same in every inertial system:

   C= 299 792 458 m/s.

2. **The laws of nature are uniform in all inertial reference frames** – there is no preferred reference frame.

In contrast to classical mechanics, in which time was assumed to be universal, special relativity shows that time intervals depend on the chosen reference frame. In a moving system, time intervals pass more slowly than in a stationary system (**time dilation)**. The length of objects is also shortened in the direction of motion (**length contraction)** (see also **Appendix 9.2**). Both phenomena are derived from the assumption of a constant speed of light for every observer, regardless of their speed.

Because space and time appear to be interdependent, Einstein introduced the concept of **space-time**: a four-dimensional structure in which space and time are united. An event in this structure has not only spatial coordinates (*x, y, z*), but also a time component (*t*), which together determine the path in space-time.

One of the most iconic outcomes of this theory is the mass-energy relation:

$$\boldsymbol{E = mc^2}$$

This formula expresses that mass and energy are equivalent. It is of fundamental importance in nuclear physics and cosmology, among other fields (see **Appendix 9.9**).

After this development, Einstein focused on further generalizing his theory to systems that do accelerate. In doing so, gravity was no longer conceived as a force, but as a manifestation of the curvature of space-time by mass and energy. This ultimately led, in 1915, to the formulation of the **general theory of relativity**.

An overview of the final formula of Einstein's field equations can be found in Chapter 2.16. The following chapters build up to this equation step by step, starting with the equivalence principle and ending with the tensor structure of space-time curvature.

2.1 The Equivalence Principle

In his quest to extend the special theory of relativity to situations involving gravity, Einstein came to a profound insight: gravity and inertia are locally physically equivalent. This insight forms the basis of the general theory of relativity.

2.1.1 Comparison with other forces

To understand the unique nature of gravity, we compare it with other fundamental forces such as the electric and magnetic forces.

Electric force

Between two charged particles with charges q_1 and q_2 , a force acts according to Coulomb's law:

$$F = k_e \frac{q_1 q_2}{r^2}$$

where *r* is the distance between the charges, and k_e is the electric constant. The resulting acceleration of a particle depends on its mass:

$$F = m_1 a_1 = k_e \frac{q_1 q_2}{r^2} \;\Rightarrow\; a_1 = k_e \frac{q_1 q_2}{m_1 r^2}$$

The acceleration therefore depends on both the charge and the mass of the particle**.**

Magnetic force

Magnetic forces also cause acceleration, depending on the charge, the speed of the particle, the orientation and strength of the magnetic field, and the mass.

Gravitational force

According to Newton, the gravitational force between two masses m_1 and m_2 is given by:

$$F = G \frac{m_1 m_2}{r^2}$$

where G is the gravitational constant and *r is* the distance between the masses.

Based on analogy with the electric force, we might expect there to be a distinction between a **gravitational mass** m_{grav} , which causes gravity, and an **inertial mass** m_{inert} , which determines how strongly an object responds to a force. In that case, the equation of motion would be:

$$F = m_{inert,1} \cdot a_1 = G \frac{m_{grav,1} \cdot m_{grav,2}}{r^2}$$

Which leads to:

$$a_1 = G \frac{m_{grav,1} \cdot m_{grav,2}}{m_{inert,1} r^2}$$

At first glance, there is no reason why $m_{inert\,,1}$ should be exactly equal to $m_{grav\,,1}$. Yet countless experiments—including **Eötvös's** precise torsion balance measurements in the 19th century—show that these two masses are always exactly equal, within the limits of measurement accuracy.

Another important difference with the electric force is that gravity is **always attractive**: there are no positive or negative masses, as there are positive and negative charges in the electromagnetic force.

Thanks to the experimentally confirmed equality $m_{grav} = m_{inert}$, the force equation simplifies to:

$$F = ma = G\frac{mM}{r^2}$$

where *M* is the mass of the attracting body (e.g., the Earth) and *m* is the mass of the test object.

The corresponding acceleration *a* of the object with mass *m* then becomes

$$a = G\frac{M}{r^2}$$

It is remarkable that the mass *m* of the falling object disappears from the equation. This means that all objects — regardless of their mass — experience the same gravitational acceleration, as long as external influences such as air resistance are neglected.

In contrast to electrical forces, where the acceleration depends on both the mass and the charge of the particle, the gravitational acceleration is therefore **independent of the mass of the test object**. This is a fundamental result that underlies Einstein's **principle** of equivalence.

2.1.2 Einstein's thought experiment

Inspired by this observation, Einstein imagined two situations:

1. A person standing on Earth experiences a gravitational acceleration of $g = 9{,}81\mathrm{m/s^2}$,
2. That same person is in a sealed rocket in empty space, moving upward at the same acceleration.

In both cases, the person feels exactly the same thing: a force pressing him against the floor. There is no local physical experiment with which he can determine the difference between these two situations. This led to the **equivalence principle**:

In a small area, gravity and acceleration are physically indistinguishable.

This principle implies that what we experience as gravity is actually a consequence of the fact that we are in an accelerating system. According to Einstein, gravity is not a force in the classical sense, but an expression of the fact that space-time is curved.

2.1.3 Influence on light

In classical mechanics, massless particles, such as photons, would not be affected by gravity. However, general relativity states that masses distort space-time, and that all objects—even light—follow this curvature. Gravity therefore also influences the path of light rays.

2.1.4 Experimental confirmation

In 1919, Arthur Eddington confirmed this effect during a solar eclipse. He observed that stars near the sun appeared to shift slightly in the sky—exactly as predicted by Einstein. The derivation of this effect follows later in the chapter on the deflection of light (Experiment 3 - Deflection of Light).

2.1.5 Key insights

- **Gravity versus other forces:** While forces such as the electric force depend on both mass and charge, gravity appears to be unique in that all objects—regardless of their mass—experience the same acceleration in a gravitational field.
- **Gravitational and inertial mass are equal:** Experiments show that the mass that *causes* gravity (gravitational mass) is equal to the mass that responds to a force (inertial mass).
- **Acceleration independent of mass:** As a result, all objects fall with the same acceleration, which is not self-evident with other forces.
- **Einstein's thought experiment:** Someone in an elevator on Earth experiences the same thing as someone in an accelerating rocket in empty space→ local equivalence between gravity and acceleration.
- **Consequence:** Gravity is no longer seen as *a force*, but as a consequence of the curvature of space-time.

2.1.6 Intuitive explanation

Imagine you are in a closed space—a room without windows, or a rocket. You feel yourself being pressed against the floor. However, you do not know whether this is because you are on Earth or because the room is moving upward at a constant acceleration. If there is no physically perceptible difference, then that difference does not *exist* locally.

According to Einstein, gravity is nothing more than the effect of moving in curved space-time.

Just as a ball on a trampoline rolls to the lowest point without anyone pushing it, every object moves along the path prescribed by curved space-time. That path is called a **geodesic line**. This is the equivalent of a 'straight line' in curved space-time.

2.2 Curvature of Space-Time

To understand the importance of the transition from Newton's classical model of gravity to Einstein's geometric model, we first approach the subject in an alternative, more intuitive way.

Imagine a particle in free space, far away from masses and without the influence of external forces. In such a situation, the particle continues to move at a constant speed and in a straight line—a principle that was described by **Galileo Galilei** around 1600.

When we imagine space-time as being made up of rectangular grid lines—a spatial frame of reference without curvature—the particle moves along a straight line through this flat grid. There is nothing to cause it to deviate from its initial direction or speed.

Einstein, however, argued that this picture changes as soon as a **large mass** is present. That mass distorts the structure of space-time, causing the "straight" lines of the grid to become **curved**. Instead of gravity acting as a force, the particle moves along these curved lines **by itself**.

The closer the particle gets to the mass, the more its path deviates from its previous straight line. Yet the particle does not feel any force: it moves freely, but follows the curvature of space. This path turns out to be a kind of "straight line" within the curvature, and is referred to later in this document as a **geodesic line**.

In general relativity, therefore, there is no gravitational force as in Newton's theory, but the effect of gravity arises from the geometry of space-time itself.

2.2.1 From force to geometry

The challenge Einstein faced was to find a mathematical description of this curvature. He sought a way to express the geometry of space-time **as a function of mass and energy**, whereby that description **would be independent of the chosen coordinate system**.

This meant that he was looking for a completely **coordinate-independent** formulation, so that the laws of nature would retain the same form in every system—a central principle in general relativity.

The effects of mass and energy on geometry would ultimately be laid down in the so-called **Einstein field equations**. These equations describe how matter curves space-time, and how that curved space-time in turn determines how matter moves.

In the coming chapters, we will follow Einstein's line of thinking step by step. In doing so, we will gradually work towards the derivation of the field equations that form the core of general relativity.

2.2.2 Independence of the Chosen Coordinate System

To determine the position of a point in space, we always need a reference—an origin from which we measure distances. A common method is to choose a **Cartesian coordinate system**, with three mutually perpendicular axes: the *x*-axis, *y*-axis, and *z*-axis.

We can describe the location of a point with coordinates (*x, y, z*), where these values represent the distances to the origin along the *x*, *y*, and *z* axes, respectively. The distance from that point to the origin is then, according to Pythagoras' theorem:

$$s = \sqrt{x^2 + y^2 + z^2}$$

When we choose a different coordinate system (with a different origin or rotation), the coordinate values change and so does *s*. Although the coordinates of individual points change during a transformation, the small distance between two nearby points — a differential element — remains invariant.

We denote this differential distance with:

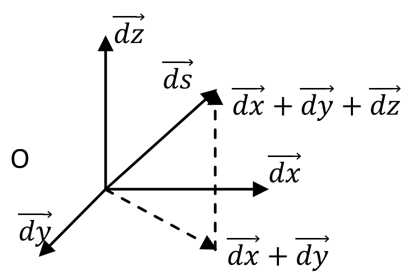
$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

This formula is applicable in an orthogonal, flat, Cartesian system. But to make it more general—also for situations in which the axes are not necessarily perpendicular—we must use a more fundamental approach via vector analysis.

2.2.3 Vector approach to distance

We can interpret the differential distance $\overrightarrow{ds}$ as the sum of three vector components:

$$\overrightarrow{ds} = \overrightarrow{dx} + \overrightarrow{dy} + \overrightarrow{dz}$$



We find the magnitude *ds* of the vector $\overrightarrow{ds}$ via the inner product $\overrightarrow{ds} \cdot \overrightarrow{ds}$:

$$ds^2 = \overrightarrow{ds} \cdot \overrightarrow{ds} \ = (\overrightarrow{dx} + \overrightarrow{dy} + \overrightarrow{dz}) \cdot (\overrightarrow{dx} + \overrightarrow{dy} + \overrightarrow{dz})$$

As a reminder, the inner product of two vectors $\vec{A}$ and $\vec{B}$ is:

$$\vec{A} \cdot \vec{B} = AB\cos(\varphi)$$

Where φ is the angle between the two vectors.

And so:

$$\overrightarrow{ds} \cdot \overrightarrow{ds} = ds^2\, cos(0) = ds^2$$

For the full inner product of $\overrightarrow{ds}$, we get:

$$ds^2 = \overrightarrow{\boldsymbol{dx}} \cdot \overrightarrow{\boldsymbol{dx}} + \overrightarrow{dx} \cdot \overrightarrow{dy} + \overrightarrow{dx} \cdot \overrightarrow{dz} + \overrightarrow{dy} \cdot \overrightarrow{dx} + \overrightarrow{\boldsymbol{dy}} \cdot \overrightarrow{\boldsymbol{dy}} + \overrightarrow{dy} \cdot \overrightarrow{dz} + \overrightarrow{dz} \cdot \overrightarrow{dx} + \overrightarrow{dz} \cdot \overrightarrow{dy} + \overrightarrow{\boldsymbol{dz}} \cdot \overrightarrow{\boldsymbol{dz}}$$

In an **orthogonal system**, the cross products such as $\overrightarrow{dx} \cdot \overrightarrow{dy}$ disappear, because the angles between the axes are 90° and $cos(90°) = 0$. In that case, we simply get:

$$ds^2 = dx^2 + dy^2 + dz^2$$

In a **non-orthogonal coordinate system**, the angles between the axes are not necessarily 90°, which means that the cross products also count. The general form then becomes:

$$ds^2 = g_{xx}dx^2 + g_{xy}dxdy + g_{xz}dxdz + g_{yx}dydx + g_{yy}dy^2 + g_{yz}dydz + g_{zx}dzdx + g_{zy}dzdy + g_{zz}dz^2 \quad (1)$$

The coefficients g_{ij} provide information about the mutual orientation of the axes, and together form the **metric tensor** g_{ij} .

2.2.4 Extension to Space-time

Einstein sought a more general formulation of the theory of relativity, applicable to a **four-dimensional coordinate system** consisting of one time axis and three space axes. In this space-time, the axes do not necessarily have to be orthogonal, and moreover, **the metric**—the way distances are measured—can differ from point to point.

The general expression for the **square of the space-time interval** is then:

$$ds^2 = \sum_{\mu}\sum_{\nu} g_{\mu\nu}\,dx^{\mu}dx^{\nu}$$

Where:

- $\mu, \nu = 0, 1, 2, 3,$
- $x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z$
- $g_{\mu\nu}$ are the components of the **four-dimensional metric tensor**.

For **dimensional consistency,** the time coordinate must have the same unit as the spatial coordinates. Therefore, time t is multiplied by the speed of light c, so that $x^0 = ct$ has the unit meter.

In **Einstein notation**, which automatically sums over repeated indices (the so-called *"dummy indices"*), this expression simplifies to:

$$ds^2 = g_{\mu\nu}\,dx^{\mu}dx^{\nu} \quad (2)$$

This metric defines the structure of space-time and determines how distances, time intervals, and causal relations are calculated at each point in the field.

2.2.4.1 Elaboration of the sum

When we write out equation (2) completely for all values of μ and ν, we get:

$$\begin{aligned} ds^2 = {} & g_{00}dx^0dx^0 + g_{01}dx^0dx^1 + g_{02}dx^0dx^2 + g_{03}dx^0dx^3 + \\ & g_{10}dx^1dx^0 + g_{11}dx^1dx^1 + g_{12}dx^1dx^2 + g_{13}dx^1dx^3 + \\ & g_{20}dx^2dx^0 + g_{21}dx^2dx^1 + g_{22}dx^2dx^2 + g_{23}dx^2dx^3 + \\ & g_{30}dx^3dx^0 + g_{31}dx^3dx^1 + g_{32}dx^3dx^2 + g_{33}dx^3dx^3 \end{aligned} \quad (3)$$

This is the four-dimensional counterpart of equation (1), which we previously set up for a three-dimensional space. (For more detailed information, see chapter 5)

2.2.4.2 Note on symmetry

The metric tensor $g_{\mu\nu}$ is **symmetric**, which means that:

$$g_{\mu\nu} = g_{\nu\mu}$$

Therefore, the tensor contains only 10 independent components instead of 16. This makes the tensor mathematically elegant and practically manageable.

2.2.5 Key insights

- The **distance between two infinitesimally close points** in space-time is not influenced by the choice of coordinate system.
- In general relativity, this distance is expressed by the **metric tensor**, which takes into account both curvature and non-orthogonality.

- **Free motion in flat space**: A particle without the influence of forces moves in a straight line at a constant speed (inertial motion).
- **Space-time as geometry**: In Einstein's view, mass distorts the structure of space-time, causing 'straight lines' (inertial trajectories) to become curved.
- **Gravity = curvature**: Instead of a force (as in Newton's theory), gravity in relativity theory is the result of the curvature of space-time.
- **Geodesics (geodesic lines)**: In this curved space, objects follow the shortest or 'straightest' paths, even though they appear curved to an external observer.
- **Einstein's challenge**: Develop a coordinate-independent mathematical description of how mass distorts space-time→ Einstein field equations.

For a more in-depth look at tensors, metrics, and their application to specific cases such as the Schwarzschild solution, please refer to **chapter** 5.

2.2.6 Intuitive explanation

Imagine:

- A billiard ball rolls across a **smooth, flat** table—it moves in a straight line.
- Now place a heavy metal ball on a **stretchy rubber mat** (such as a trampoline), and a **curvature** is created.
- If you roll a smaller ball onto the mat now, that ball will **be deflected by the deformation**, even though no one is pushing it.

That is gravity in Einstein's view.

According to Newton, gravity is a *force at a distance*. According to Einstein, there is **no force**: objects move in straight lines, but those straight lines lie on a **curved surface**.

In this sense, a falling apple is not attracted, but simply follows the shortest route through a distorted space-time.

2.3 Covariant and Contravariant Vectors and Dual Vectors

In general relativity, the terms **contravariant** and **covariant** occur regularly. In this section, we explain these concepts and show how they arise from the way vectors and fields transform under a change of coordinate system.

As discussed earlier, physical quantities—such as vectors, tensors, and fields**—must be independent of the chosen coordinate system**. When transitioning to another system (e.g., via rotation or translation), the physical properties remain unchanged, but their components change in a specific way: they **transform according to well-defined rules**, depending on the type of object (covariant or contravariant).

2.3.1 Scalar quantities, vectors, and fields

A **scalar quantity**, such as temperature, has a value at every location but no direction. A collection of scalars across space forms a **scalar field**.

When such a field exhibits a direction-dependent change (for example, a temperature increase in a certain direction), we can take its derivative. This derivative **behaves like a vector**, and in this specific case we refer to it as a **dual vector**.

A **dual vector** depends on the chosen coordinate system: during a transformation, the components of the vector change in such a way that the whole remains physically consistent. Because these components *transform along* with the coordinate system, they are called **covariant**.

A "normal" vector (such as velocity or acceleration) reacts differently: when the coordinate system changes, the underlying vectors remain physically the same, but their components change in **the opposite direction** to the basis vectors. Such vectors are called **contravariant**.

2.3.1.1 Notation and definitions

To distinguish between the two types of vectors, the following notation is conventionally used:

- A **contravariant vector** has a **superscript**:

$$A^\mu$$

- A **covariant vector** has a **subscript**:

$$A_\mu$$

These are connected via the **metric tensor** $g_{\mu\nu}$ according to the relation:

$$A_\mu = g_{\mu\nu} A^\nu$$

The contraction of a contravariant vector with its covariant counterpart yields a **scalar invariant**:

$$A_\mu A^\mu = I$$

This expression means that the inner product of a vector with its dual (or 'reduced') version results in a quantity I that **remains invariant under coordinate transformations**. This number$I$ can be interpreted as the norm or the square of the distance in space-time, depending on the sign:

- Time-like**:** $I < 0$
- Spatial**:** $I > 0$
- Light-like**:** $I = 0$

This classification clarifies how the **metric tensor** plays a key role: it determines not only how components of vectors are transformed, but also how distances, lengths, and causal structures are defined in curved space-time.

2.3.2 Transformations between coordinate systems

Suppose we are working in a coordinate system with coordinates x^m (where m = 0, 1, 2, 3), and we switch to a new coordinate system with coordinates y^n. The relationship between the two systems is then given by:

$$y^n = \frac{\partial y^n}{\partial x^0} x^0 + \frac{\partial y^n}{\partial x^1} x^1 + \frac{\partial y^n}{\partial x^2} x^2 + \frac{\partial y^n}{\partial x^3} x^3$$

In Einstein notation, which automatically sums over repeated indices (from 0 to 3), this becomes:

$$y^n = \frac{\partial y^n}{\partial x^m} x^m$$

2.3.2.1 Example: derivative of a scalar function

Consider a scalar function φ . The differential is:

$$d\varphi = \frac{\partial \varphi}{\partial x^m} dx^m$$

Written out in full:

$$d\varphi = \frac{\partial \varphi}{\partial x^0} dx^0 + \frac{\partial \varphi}{\partial x^1} dx^1 + \frac{\partial \varphi}{\partial x^2} dx^2 + \frac{\partial \varphi}{\partial x^3} dx^3$$

In the new coordinate system y^n , we use the chain rule to transform the components of the derivative:

$$\frac{d\varphi}{dy^n} = \frac{\partial\varphi}{\partial x^m}\frac{dx^m}{dy^n}$$

It follows that the components transform as:

$$A_n^{(y)} = \frac{dx^m}{dy^n} B_m^{(x)} \tag{1}$$

where:

- $A_n^{(y)} = \frac{d\varphi}{dy^n}$: the covariant vector in the y-system,
- $B_m^{(x)} = \frac{\partial\varphi}{\partial x^m}$: the covariant vector in the x-system.

This is a **covariant transformation.**

2.3.2.1.1 Fully written out (matrix form)

In matrix form, equation (1) becomes:

$$\begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix}_y = \begin{pmatrix} \frac{dx^0}{dy^0} & \frac{dx^1}{dy^0} & \frac{dx^2}{dy^0} & \frac{dx^3}{dy^0} \\ \frac{dx^0}{dy^1} & \frac{dx^1}{dy^1} & \frac{dx^2}{dy^1} & \frac{dx^3}{dy^1} \\ \frac{dx^0}{dy^2} & \frac{dx^1}{dy^2} & \frac{dx^2}{dy^2} & \frac{dx^3}{dy^2} \\ \frac{dx^0}{dy^3} & \frac{dx^1}{dy^3} & \frac{dx^2}{dy^3} & \frac{dx^3}{dy^3} \end{pmatrix} \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix}_x$$

2.3.2.2 Contravariant transformation

For contravariant vectors, the transformation formula is reversed:

$$W_{(y)}^n = \frac{dy^n}{dx^m} B_{(x)}^m$$

Fully written out in matrix form:

$$\begin{pmatrix} W^0 \\ W^1 \\ W^2 \\ W^3 \end{pmatrix}_y = \begin{pmatrix} \frac{dy^0}{dx^0} & \frac{dy^0}{dx^1} & \frac{dy^0}{dx^2} & \frac{dy^0}{dx^3} \\ \frac{dy^1}{dx^0} & \frac{dy^1}{dx^1} & \frac{dy^1}{dx^2} & \frac{dy^1}{dx^3} \\ \frac{dy^2}{dx^0} & \frac{dy^2}{dx^1} & \frac{dy^2}{dx^2} & \frac{dy^2}{dx^3} \\ \frac{dy^3}{dx^0} & \frac{dy^3}{dx^1} & \frac{dy^3}{dx^2} & \frac{dy^3}{dx^3} \end{pmatrix} \begin{pmatrix} B^0 \\ B^1 \\ B^2 \\ B^3 \end{pmatrix}_x$$

2.3.3 Transformation behavior of basis vectors

In tensor calculus, it is important to understand not only how the **components** of a vector change during coordinate transformation, but also how the corresponding **basis vectors** themselves transform.

When changing the coordinate system from x^m to y^n , the corresponding basis vectors are respectively:

- $\vec{e}_m = \frac{\partial}{\partial x^m}$
- $\vec{f}_n = \frac{\partial}{\partial y^n}$

The relationship between basis vectors in different coordinates follows from the chain rule of differential calculus:

$$\frac{\partial}{\partial x^m} = \frac{\partial y^n}{\partial x^m}\frac{\partial}{\partial y^n} \quad \Rightarrow \quad \vec{e}_m = \frac{\partial y^n}{\partial x^m}\vec{f}_n$$

It follows that the **basis vectors transform covariantly**: they change **along** with the coordinate system. The components of contravariant vectors must therefore adapt in **the opposite direction** to keep the whole physically invariant.

2.3.3.1 Note on Einstein notation

Einstein notation uses repeated indices (so-called *dummy indices*), which are automatically summed over the values 0 to 3:

$$A^\mu B_\mu = \sum_{\mu=0}^{3} A^\mu B_\mu$$

In this section, many equations have been deliberately **written out explicitly** to clarify the meaning of this notation. In later chapters, we will more often use the abbreviated Einstein notation for compactness.

2.3.4 Key insights

- **Scalars versus vectors**:
    - A **scalar quantity** (such as temperature) does not change under a coordinate transformation.
    - A **vector** has both direction and magnitude. The components of a vector do change under transformation, depending on the type of vector.
- **Contravariant vectors** (such as position or velocity vectors W^n):
    - Transform **opposite to the basis vectors** to keep the vector physically constant.
    - Transformation formula:

$$W^n_{(y)} = \frac{dy^n}{dx^m} B^m_{(x)}$$

- **Covariant vectors** (such as dual vectors A_n):
    - Transform **along with the coordinate system.**
    - Transformation formula:

$$A^{(y)}_n = \frac{dx^m}{dy^n} B^{(x)}_m$$

- **Duality**:
    - Mathematically speaking, covariant vectors are **linear functions on vectors**; they belong to the so-called **dual vector space**.
- **Converting between covariant and contravariant:**
    - Using the metric tensor $g_{\mu\nu}$, we can convert contravariant and covariant vectors into each other:

$$A_\mu = g_{\mu\nu} A^\nu, \quad \text{and reversed:} \quad A^\mu = g^{\mu\nu} A_\nu$$

2.3.5 Intuitive explanation

Imagine you are standing on a hill and measuring the slope in different directions:

- If you look north, you see an increase of, for example, 10 meters per kilometer.
- If you now rotate your coordinate system (for example, by turning a compass), the numbers you use to describe that slope will change—but **the hill itself will not change, of course**.

This is precisely the essence of **tensor transformations**:
The direction of a vector (e.g., a slope or force) remains physically the same, but the **coordinates in which you express that vector change** with the system.

- For **contravariant vectors** (such as position or velocity), the components change in the **opposite direction** to the change in the coordinate system. They *correct,* as it were, for the rotation of the axes.
- For **covariant vectors** (such as the gradient of a scalar field), the components change **along** with the new system. Think of how the direction of a slope is related to how you measure space.

The **metric** acts as a kind *of converter* between the two types of vectors. You can think of the metric as a ruler that **measures differently** in each direction, depending on the local curvature of space-time.

Comparison table

Property	Contravariant	Covariant
Index position	Above A^{μ}	Below A_{μ}
Transforms...	Opposite to base	Along with base
Example	Position, velocity	Gradient, differential
Origin	Direction in space	Directional derivative of a scalar field

2.4 Covariant and contravariant transformations of tensors

In general relativity—and more broadly in tensor analysis**—covariant**, **contravariant**, and **mixed tensors** play a central role. The way in which these tensors transform under a change of coordinate system is essential for formulating physical laws **in a coordinate-independent manner**. In this section, we discuss the transformation properties of the different types of tensors.

The transformation rules discussed here are a direct extension of the rules for vectors from the previous section.

2.4.1 Covariant Tensors

A **covariant tensor** has one or more subscripts, such as T_{mn} , and can be constructed from the product of covariant vectors A_m and B_n

The transformation of a covariant tensor from a coordinate system *x* to a new system *y* proceeds as follows:

$$T_{mn}^{(y)} = A_m^{(y)} B_n^{(y)} = \frac{dx^r}{dy^m} A_r^{(x)} \frac{dx^s}{dy^n} B_s^{(x)} = \frac{dx^r}{dy^m} \frac{dx^s}{dy^n} A_r^{(x)} B_s^{(x)} = \frac{dx^r}{dy^m} \frac{dx^s}{dy^n} T_{rs}^{(x)}$$

The result of the transformation of T_{rs} to T_{mn} is then given by:

$$T_{mn}^{(y)} = \frac{dx^r}{dy^m}\frac{dx^s}{dy^n}T_{rs}^{(x)}$$

Where:

- $T_{mn}^{(y)}$ the covariant tensor in the new coordinate system *y*,
- The factors $\frac{dx^r}{dy^m}$ and $\frac{dx^s}{dy^n}$ are the Jacobian components of the transformation from *y* to *x*,
- $T_{rs}^{(x)}$ is the original covariant tensor in the old system.

2.4.2 Contravariant Tensors

A **contravariant tensor** has one or more superscripts, such as T^{mn} , and can be constructed from contravariant vectors A^m and B^n .

The transformation is opposite to that of the covariant tensor:

$$T_{(y)}^{mn} = A_{(y)}^m B_{(y)}^n = \frac{dy^m}{dx^r}A_{(x)}^r \frac{dy^n}{dx^s}B_{(x)}^s = \frac{dy^m}{dx^r}\frac{dy^n}{dx^s}A_{(x)}^r B_{(x)}^s = \frac{dy^m}{dx^r}\frac{dy^n}{dx^s}T_{(x)}^{rs}$$

The result of the transformation of T^{rs} to T^{mn} is then given by:

$$T_{(y)}^{mn} = \frac{dy^m}{dx^r}\frac{dy^n}{dx^s}T_{(x)}^{rs}$$

This formula indicates how the components of a contravariant tensor adapt to a change of basis.

2.4.3 Mixed Tensors

A **mixed tensor** contains both upper and lower indices, for example $T^m{}_n$. Such a tensor can arise, for example, from the product of a contravariant vector A^m and a covariant vector B_n .

The corresponding transformation formula is:

$$T^m{}_n(y) = A_{(y)}^m B_n^{(y)} = \frac{dy^m}{dx^r}A_{(x)}^r \frac{dx^s}{dy^n}B_s^{(x)} = \frac{dy^m}{dx^r}\frac{dx^s}{dy^n}A_{(x)}^r B_s^{(x)} = \frac{dy^m}{dx^r}\frac{dx^s}{dy^n}T^r{}_s(x)$$

So, the **transformation** of a **mixed** tensor is:

$$T^m{}_n(y) = \frac{dy^m}{dx^r}\frac{dx^s}{dy^n}T^r{}_s(x)$$

This mix of derivatives reflects the combined behavior of the different index types.

2.4.4 Key insights

- A **tensor** is characterized by its rank (number of indices) and the type of indices (upper or lower).
- Tensors are the natural language for formulating physical laws that are independent of the chosen coordinate system.
- The **transformation properties** of a tensor guarantee that it retains its meaning under coordinate changes.

Rank and Notation

- A **tensor of rank 0** is a **scalar quantity**, such as temperature or mass. It does not change under coordinate transformations.
- A **vector** is a **tensor of rank 1** and can occur in two forms:
  - **Contravariant**: denoted with a superscript, for example V^m
  - **Covariant**: denoted with a subscript, for example V_m
- A **tensor of rank 2** has several forms:
  - **Fully covariant**: $T_{\mu\nu}$,
  - **Fully contravariant**: $T^{\mu\nu}$,
  - **Mixed**: $T^{\mu}{}_{\nu}$, etc.

Transformation properties

A tensor is defined by the way its components transform under a change of coordinate system. These transformation rules ensure that the tensors retain their physical meaning, regardless of the chosen system:

- **Covariant components** (lower indices, e.g.,$T_{\mu\nu}$) transform with the derivative from the old to the new coordinates.
- **Contravariant components** (upper indices, e.g.,$T^{\mu\nu}$) transform with the derivative from the new to the old coordinates.
- **Mixed tensors** combine both rules (e.g.,T_{ν}^{μ}), depending on the position of the indices.

An important example is the **metric tensor** $= g^{\mu\nu}$, which allows us to raise or lower indices via:

$$T^{\mu} = g^{\mu\nu} T_{\nu}$$

This ability to manipulate indices makes it easy to switch between covariant and contravariant descriptions.

Physical Relevance

The fundamental equations of physics—such as Einstein's field equations in general relativity—are formulated in terms of tensors. This makes them **invariant under coordinate transformations**, which is an essential feature of any **covariant theory**. This guarantees that physical laws retain the same form regardless of the chosen coordinate system, and that the underlying geometry remains consistently described.

2.4.5 Intuitive explanation

You can compare tensor transformations to redrawing a map:

- Imagine a contour map with hills, valleys, and wind directions.
- You rotate the map 30°, but the hills remain where they **are—only the coordinates in which you describe them change**.

Tensors behave like **measurable structures in that world**:

- A vector arrow on the map (e.g., wind direction) gets **new coordinates** after the rotation, so that the **direction remains physically the same**.
- A gradient (e.g., the slope of the landscape) still points upward, but you now describe it with **different components**—depending on the new axes.

This is how tensors behave under transformations: **their geometric or physical meaning remains the same**, but the components change depending on the chosen coordinate system.

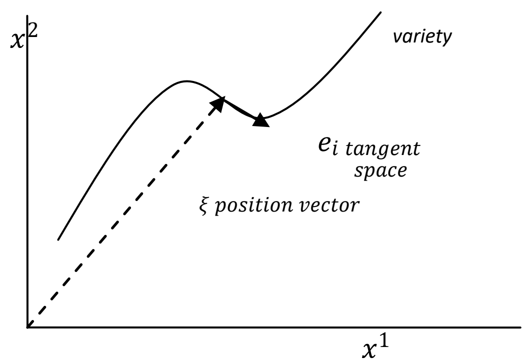
Overview of transformations

Tensor type	Index notation	Transforms as...
Scalar	ϕ	Remains unchanged
Contravariant vector	V^{μ}	$\frac{\partial y^{\mu}}{\partial x^{\nu}} V^{\mu}$
Covariant vector	V_{μ}	$\frac{\partial x^{\nu}}{\partial y^{\mu}} V_{\mu}$
Covariant tensor	$T_{\mu\nu}$	Twice the covariant rule
Contravariant tensor	$T^{\mu\nu}$	Twice the contravariant rule
Mixed tensor	$T^{\mu}{}_{\nu}$	Mix of both

2.5 Derivation of the Christoffel symbol and the covariant derivative

In order to describe gravity as a geometric phenomenon, Einstein had to find a way to mathematically capture the **curvature of space-time**. Instead of forces, general relativity introduces a structure on space-time itself, in which the **Christoffel symbol** plays a central role. This symbol describes how basis vectors change and forms the basis of the **covariant derivative,** which is necessary to differentiate consistently in curved space.

2.5.1 Basic definition of the Christoffel symbol



Consider a coordinate system x^i with an associated **position vector** $\vec{\xi}(x^i)$, pronounced as *'ksi'*, which represents a spatial manifold. We define the basis vectors in the tangent space as the partial derivatives of $\vec{\xi}$:

$$\vec{e}_i = \frac{\partial \vec{\xi}}{\partial x^i}$$

The derivative of this basis vector with respect to another coordinate x^j indicates how the direction of the basis vector changes in space:

$$\frac{\partial \vec{e}_i}{\partial x^j} = \frac{\partial^2 \vec{\xi}}{\partial x^i \partial x^j}$$

This second derivative can be expressed as a linear combination of the basis vectors themselves:

$$\frac{\partial \vec{e}_i}{\partial x^j} = \Gamma_{ij}^k \vec{e}_k \tag{1}$$

Here, Γ_{ij}^k is the **Christoffel symbol of the second kind**. This object describes how the basis vectors change, and thus the **curvature of space**. If this derivative is zero, the direction of the basis vector does not change and the space is flat.

2.5.1.1 Vectorial Interpretation of Direction Change

The basis vectors $\vec{e}_i$ belong to the **tangent space** at a point of the manifold. The derivative from equation (1) tells us how this basis changes in the direction of x^j. If $\frac{\partial \vec{e}_i}{\partial x^j} \neq 0$, the space is **curved**.

Written out in full, equation (1) has the form:

$$\frac{\partial \vec{e}_i}{\partial x^j} = \Gamma_{ij}^k \vec{e}_k = \Gamma_{ij}^0 \vec{e}_0 + \Gamma_{ij}^1 \vec{e}_1 + \Gamma_{ij}^2 \vec{e}_2 + \Gamma_{ij}^3 \vec{e}_3$$

(From here on, we omit the vector sign for simplicity.)

2.5.1.2 Derivation of the Christoffel symbol

Using the **duality of basis vectors**, we take the inner product with the dual basis vectore^k :

$$e^k e_k = 1 \tag{2}$$

By multiplying both sides of equation (1) by e^k , we obtain:

$$\Gamma_{ij}^k = e^k \left(\frac{\partial e_i}{\partial x^j}\right) \tag{3}$$

This gives a direct definition of the Christoffel symbol.

2.5.1.3 Symmetry of the lower indices

Because in a smooth manifold the order of differentiation does not matter (they are commutative), $(\partial_i \partial_j = \partial_j \partial_i)$, the following applies:

$$\frac{\partial e_i}{\partial x^j} = \frac{\partial e_j}{\partial x^i} \Longrightarrow e^k \frac{\partial e_i}{\partial x^j} = e^k \frac{\partial e_j}{\partial x^i} \Longrightarrow \Gamma_{ij}^k = \Gamma_{ji}^k \tag{4}$$

The Christoffel symbol is therefore symmetric in the lower indices $\Gamma_{ij}^k = \Gamma_{ji}^k$.

2.5.1.4 Derivation via coordinate transformation

Consider again:

$$e_k = \frac{\partial \xi}{\partial x^k} \Longrightarrow e^k = \frac{1}{e_k} = \frac{\partial x^k}{\partial \xi} \tag{5}$$

Substitution in equation (1) yields:

$$\Gamma_{ij}^k = e^k \left(\frac{\partial^2 \xi}{\partial x^i \partial x^j}\right) \tag{6}$$

Or, written in reverse:

$$\Gamma_{ij}^k = \frac{\partial x^k}{\partial \xi} \cdot \frac{\partial^2 \xi}{\partial x^i \partial x^j}$$

This expression shows that the Christoffel symbol is constructed from **second derivatives of the coordinates**, and is therefore directly related to the **geometry of space-time**.

2.5.1.5 Link to the metric tensor

The **metric tensor** g_{ik} is defined as the inner product of the basis vectors:

$$g_{ik} = e_i . e_k = \left(g^{ik}\right)^{-1} \tag{7}$$

Using the inverse metric g^{ik} , we can convert basis vectors between each other:

$$e_i . \frac{1}{e^k} = \left(g^{ik}\right)^{-1}$$

$$e^k = g^{ik} e_i \tag{8}$$

2.5.1.6 Summary

- The **Christoffel symbol** Γ_{ij}^k describes how basis vectors change in a curved space.
- It plays a central role in the definition of the **covariant derivative**, which will be discussed in the next chapter.
- The **symmetry** $\Gamma_{ij}^k = \Gamma_{ji}^k$ follows from the commutativity of partial derivatives.
- The Christoffel symbol can be expressed both in terms of **coordinate derivatives** and in terms of the **metric tensor**, and is thus fundamentally linked to the structure of space-time.

2.5.2 Covariant Derivative

The **covariant derivative** is an extension of the concept of the ordinary derivative in flat (Euclidean) space. In the context of general relativity, this derivative must be adapted so that it is valid in **curved space-time**.

Einstein required that his theory be **covariant**, which means that physical laws retain the same form in every coordinate system. In particular, if the derivative of a tensor is zero in one system, then it must also be zero in every other coordinate system.

To guarantee this, we define the **covariant derivative ∇** , which corrects the ordinary derivative with additional terms. This derivative satisfies the requirement:

$$\nabla g_{mn} = 0$$

This condition defines the unique torsion-free connection ∇ , also known as the **Levi-Civita connection**.

2.5.2.1 Metrics and derivatives

We start with the metric tensor (7):

$$g_{mn} = e_m \cdot e_n \tag{9}$$

Take the ordinary derivative with respect tox^s :

$$\frac{\partial g_{mn}}{\partial \mathrm{x}^{\mathrm{s}}} = \frac{\partial(e_m.e_n)}{\partial \mathrm{x}^{\mathrm{s}}} = e_m \frac{\partial e_n}{\partial \mathrm{x}^{\mathrm{s}}} + e_n \frac{\partial e_m}{\partial \mathrm{x}^{\mathrm{s}}} \tag{10}$$

Due to the symmetry derived earlier (see equation 4), we can write:

$$\frac{\partial g_{mn}}{\partial \mathrm{x}^{\mathrm{s}}} = e_m \frac{\partial e_n}{\partial \mathrm{x}^{\mathrm{s}}} + e_n \frac{\partial e_m}{\partial \mathrm{x}^{\mathrm{s}}}$$

$$\Rightarrow \frac{\partial g_{mn}}{\partial \mathrm{x}^{\mathrm{s}}} = e_m \frac{\partial e_s}{\partial \mathrm{x}^{\mathrm{n}}} + e_n \frac{\partial e_s}{\partial \mathrm{x}^{\mathrm{m}}} \tag{11}$$

If we move these terms to one side of the equation, we get:

$$\frac{\partial g_{mn}}{\partial \mathrm{x}^{\mathrm{s}}} - e_m \frac{\partial e_s}{\partial \mathrm{x}^{\mathrm{n}}} - e_n \frac{\partial e_s}{\partial \mathrm{x}^{\mathrm{m}}} = 0 \tag{12}$$

2.5.2.2 Definition of the covariant derivative

This relationship motivates the definition of the **covariant derivative of the metric**:

$$\nabla_{\mathrm{s}} g_{mn} = \frac{\partial g_{mn}}{\partial \mathrm{x}^{\mathrm{s}}} - e_m \frac{\partial e_s}{\partial \mathrm{x}^{\mathrm{n}}} - e_n \frac{\partial e_s}{\partial \mathrm{x}^{\mathrm{m}}} = 0 \tag{13}$$

We now express the tangent space derivatives in terms of **Christoffel symbols**. From the previous chapter, we know:

$$\Gamma_{sn}^{t} = e^t \frac{\partial e_s}{\partial x^n} \quad \text{and} \quad g_{mt} = e_m \cdot e_t$$

This makes equation (13):

$$\nabla_{\mathrm{s}} g_{mn} = \frac{\partial g_{mn}}{\partial \mathrm{x}^{\mathrm{s}}} - e_m \frac{\boldsymbol{\partial e_s}}{\boldsymbol{\partial \mathrm{x}^{\mathrm{n}}}} \boldsymbol{e^t} e_t - e_n \frac{\boldsymbol{\partial e_s}}{\boldsymbol{\partial \mathrm{x}^{\mathrm{m}}}} \boldsymbol{e^t} e_t = 0 \tag{14}$$

Here we obtain the **covariant derivative** of the metric tensor, expressed in terms of the normal derivative, corrected by two terms that are products of the metric tensor and the corresponding Christoffel symbol:

$$\boldsymbol{\nabla_{s} g_{mn} = \frac{\partial g_{mn}}{\partial \mathrm{x}^{\mathrm{s}}} - g_{mt}\Gamma_{sn}^{t} - g_{nt}\Gamma_{sm}^{t} = 0} \tag{15}$$

2.5.2.3 Cyclic permutation

Applying the same logic to permutations of the indices, we obtain:

$$\nabla_{\mathrm{m}} g_{ns} = \frac{\partial g_{ns}}{\partial \mathrm{x}^{\mathrm{m}}} - g_{nt}\Gamma_{ms}^{t} - g_{st}\Gamma_{mn}^{t} = 0 \tag{16}$$

$$\nabla_{\mathrm{n}} g_{sm} = \frac{\partial g_{sm}}{\partial \mathrm{x}^{\mathrm{n}}} - g_{st}\Gamma_{nm}^{t} - g_{mt}\Gamma_{ns}^{t} = 0 \tag{17}$$

Now we perform the following operation: (17) + (16) - (15), taking into account the symmetry mentioned in equation (4), that $\Gamma_{ij}^{k} = \Gamma_{ji}^{k}$, resulting in:

$$\frac{\partial g_{sm}}{\partial \mathrm{x}^{\mathrm{n}}} + \frac{\partial g_{ns}}{\partial \mathrm{x}^{\mathrm{m}}} - \frac{\partial g_{mn}}{\partial \mathrm{x}^{\mathrm{s}}} - 2g_{st}\Gamma_{nm}^{t} = 0 \tag{18}$$

$$g_{st}\Gamma^t_{nm} = \frac{1}{2}\left(\frac{\partial g_{sm}}{\partial \mathrm{x}^{\mathrm{n}}} + \frac{\partial g_{ns}}{\partial \mathrm{x}^{\mathrm{m}}} - \frac{\partial g_{mn}}{\partial \mathrm{x}^{\mathrm{s}}}\right) \tag{19}$$

2.5.2.4 Christoffel symbol via metric

We isolate Γ^t_{nm} by multiplying by the inverse metric g^{st} :

$$\boldsymbol{\Gamma^t_{nm} = \frac{1}{2} g^{st}\left(\frac{\partial g_{sm}}{\partial \mathrm{x}^{\mathrm{n}}} + \frac{\partial g_{ns}}{\partial \mathrm{x}^{\mathrm{m}}} - \frac{\partial g_{mn}}{\partial \mathrm{x}^{\mathrm{s}}}\right)} \tag{20}$$

This expression gives the **Christoffel symbols** as a function of the **metric tensor and its first derivatives**.

2.5.2.5 Remarks

2.5.2.5.1 Covariance of the metric

We confirm that the covariant derivative of the metric is indeed zero (see equation 8):

$$\nabla \mathrm{A}_{\mu} = \mathrm{g}_{\mu\nu} \nabla \mathrm{A}^{\nu} \tag{20a}$$

Using:

$$\mathrm{A}_{\mu} = \mathrm{g}_{\mu\nu} \mathrm{A}^{\nu}$$

and the Leibniz rule (chain rule):

$$\nabla \mathrm{A}_{\mu} = \nabla(\mathrm{g}_{\mu\nu} \mathrm{A}^{\nu}) = \mathrm{g}_{\mu\nu} \nabla \mathrm{A}^{\nu} + \mathrm{A}^{\nu} \nabla \mathrm{g}_{\mu\nu} \tag{20b}$$

Both (20a) and (20b) must give the same result, so:

$$\mathrm{g}_{\mu\nu} \nabla \mathrm{A}^{\nu} = \mathrm{g}_{\mu\nu} \nabla \mathrm{A}^{\nu} + \mathrm{A}^{\nu} \nabla \mathrm{g}_{\mu\nu}$$

Then:

$$\mathrm{A}^{\nu} \nabla \mathrm{g}_{\mu\nu} = 0$$

Since

$$\mathrm{A}^{\nu} \neq 0 \text{ dan is } \nabla \mathrm{g}_{\mu\nu} = 0$$

It follows that the **covariant derivative of the metric is zero**, which is a fundamental property of the Levi-Civita connection.

2.5.2.5.2 Transformation rule of vector components

Consider a vector:

$$\vec{V} = V^m \vec{e}_m$$

The component in the direction of the *n*-axis is:

$$V_n = \vec{V} \cdot \vec{e}_n$$

$$V_n = V^m(\vec{e}_m \cdot \vec{e}_n)$$

As we know:

$$\mathrm{g}_{\mathrm{mn}} = \vec{e}_m \cdot \vec{e}_n = \mathrm{g}_{\mathrm{nm}}$$

So:

$$\boldsymbol{V_n = \mathrm{g}_{\mathrm{nm}} V^m} \tag{20c}$$

Conversely, via the inverse metric:

$$\mathrm{g}_{\mathrm{nm}} = \frac{1}{\mathrm{g}^{\mathrm{mn}}}$$

$$\boldsymbol{V^m = \mathrm{g}^{\mathrm{mn}} V_n} \tag{20d}$$

2.5.2.6 Covariant Derivative for a Contravariant Vector

We now want to calculate the **covariant derivative** of a **contravariant vector** field V^m. In flat space, this would simply be the ordinary partial derivative. In curved space-time, however, we must take into account the fact that the basis vectors themselves can also vary from point to point.

2.5.2.6.1 Starting point: vector in component form

We consider the vector $\vec{V}$ as a linear combination of basis vectors $\vec{e}_m$:

$$\vec{V} = V^m \vec{e}_m \tag{21}$$

The derivative of $\vec{V}$ with respect to a coordinate x^l is:

$$\frac{\partial \vec{V}}{\partial x^l} = \frac{\partial V^m}{\partial x^l} \vec{e}_m + V^m \frac{\partial \vec{e}_m}{\partial x^l} \tag{22}$$

2.5.2.6.2 Connection with the Christoffel symbol

From previous work (equation 1), we know that the derivative of the basis vector is expressed using the **Christoffel symbol**:

$$\frac{\partial \vec{e}_m}{\partial x^l} = \Gamma^k_{ml} \vec{e}_k \tag{23}$$

Substitution in equation (22) yields:

$$\frac{\partial \vec{V}}{\partial x^l} = \frac{\partial V^m}{\partial x^l} \vec{e}_m + V^m \Gamma^k_{ml} \vec{e}_k \tag{24}$$

The sum over the indices *m* and *k* uses **Einstein notation**. We may rename the **dummy indices** (see note below) and rewrite the second term by *m*→ *γ* and *k*→ *m*:

$$\frac{\partial \vec{V}}{\partial x^l} = \frac{\partial V^m}{\partial x^l} \vec{e}_m + V^\gamma \Gamma^m_{l\gamma} \vec{e}_m$$

$$\frac{\partial \vec{V}}{\partial x^l} = \left(\frac{\partial V^m}{\partial x^l} + V^\gamma \Gamma^m_{l\gamma}\right) \vec{e}_m \tag{25}$$

2.5.2.6.3 Definition of the covariant derivative

This immediately gives the definition of the **covariant derivative** of the contravariant vector V^m ;

$$\boldsymbol{\nabla_l V^m = \frac{\partial V^m}{\partial \mathrm{x}^l} + \Gamma^m_{l\gamma} V^\gamma} \tag{26}$$

The extra term (with the Christoffel symbol) corrects for the fact that the basis vectors themselves change in a curved space. The covariant derivative $\nabla_1 V^m$ is therefore **tensor-like in nature** and transforms correctly under coordinate changes.

2.5.2.6.4 Note: dummy indices

In Einstein notation, we are free to choose how to name the **dummy index**, as long as this index is summed in the product. For example:

$$V^\mu \mathrm{A}_\mu = V^0 \mathrm{A}_0 + V^1 \mathrm{A}_1 + V^2 \mathrm{A}_2 + V^3 \mathrm{A}_3$$

Whether we name the index μ, γ or k makes no difference to the end result. The index merely acts as **a placeholder** for the summation over the dimensions.

2.5.2.6.5 In summary

- The covariant derivative of a contravariant vector V^m is:

$$\nabla_l V^m = \frac{\partial V^m}{\partial \mathrm{x}^l} + \Gamma^m_{l\gamma} V^\gamma$$

- This formula corrects the ordinary derivative with a term that reflects the curvature of space-time via the Christoffel symbol.
- The result is a **tensor** of the same rank as the original vector.

2.5.2.7 Covariant Derivative for a Covariant Vector

We will now examine how the **covariant derivative** works for a **covariant vector** B_μ . We will use the **scalar product** of a contravariant vector A^μ and a covariant vector B_μ , and then apply the derivative rules.

2.5.2.7.1 Starting point: product rule on scalar quantity

Take the scalar product $A^\mu B_\mu$. The covariant derivative of this product is

$$\nabla_\alpha \left(A^\mu B_\mu\right) = (\nabla_\alpha A^\mu) B_\mu + A^\mu \left(\nabla_\alpha B_\mu\right) \tag{27}$$

Substitute the expression for $\nabla_\alpha A^\mu$ from earlier work:

$$\nabla_\alpha A^\mu = \frac{\partial A^\mu}{\partial \mathrm{x}^\alpha} + \Gamma^\mu_{\alpha\nu} A^\nu$$

Then equation (27) becomes:

$$\nabla_\alpha\left(A^\mu B_\mu\right)=\left(\frac{\partial A^\mu}{\partial \mathrm{x}^\alpha}+\Gamma_{\alpha\nu}^\mu A^\nu\right)B_\mu+A^\mu\left(\nabla_\alpha B_\mu\right) \tag{28}$$

2.5.2.7.2 Property of scalars

Because the scalar product $A^\mu B_\mu$ is a **scalar**, the covariant derivative is equal to the ordinary derivative:

$$\nabla_\alpha\left(A^\mu B_\mu\right)=\frac{\partial\left(A^\mu B_\mu\right)}{\partial \mathrm{x}^\alpha}=\frac{\partial A^\mu}{\partial \mathrm{x}^\alpha}B_\mu+A^\mu\frac{\partial B_\mu}{\partial \mathrm{x}^\alpha} \tag{29}$$

2.5.2.7.3 Comparison of both expressions

By comparing the right-hand sides of (28) and (29):

$$\cancel{\frac{\partial A^\mu}{\partial \mathrm{x}^\alpha}}B_\mu+A^\mu\frac{\partial B_\mu}{\partial \mathrm{x}^\alpha}=\left(\cancel{\frac{\partial A^\mu}{\partial \mathrm{x}^\alpha}}+\Gamma_{\alpha\nu}^\mu A^\nu\right)B_\mu+A^\mu\left(\nabla_\alpha B_\mu\right) \tag{30}$$

Now we rewrite the indices in the second terms on both sides to clean up the equation. Rename $\mu \to \sigma$ and $\nu \to \mu$ in the last term on the right-hand side. Then we are left with:

$$A^\mu\left[-\frac{\partial B_\mu}{\partial \mathrm{x}^\alpha}+\Gamma_{\alpha\mu}^\sigma B_\sigma+\left(\nabla_\alpha B_\mu\right)\right]=0 \tag{31}$$

Since this equation must hold for every A^μ, it follows that:

$$\boldsymbol{\nabla_\alpha B_\mu=\frac{\partial B_\mu}{\partial \mathrm{x}^\alpha}-\Gamma_{\alpha\mu}^\sigma B_\sigma} \tag{32}$$

2.5.2.7.4 Definition

This is the **covariant derivative of a covariant vector** B_μ . The formula is analogous to that of contravariant vectors, but the Christoffel symbol now has a **minus sign** and the index positions are swapped:

For V^m :

$$\nabla_l V^m=\frac{\partial V^m}{\partial \mathrm{x}^l}+\Gamma_{l\gamma}^m V^\gamma$$

- For V^m : $\nabla_l V^m=\frac{\partial V^m}{\partial \mathrm{x}^l}+\Gamma_{l\gamma}^m V^\gamma$
- For B_μ : $\nabla_\alpha B_\mu=\frac{\partial B_\mu}{\partial \mathrm{x}^\alpha}-\Gamma_{\alpha\mu}^\sigma B_\sigma$

2.5.2.7.5 In summary

- The covariant derivative of a covariant vectorB_μ is:

$$\nabla_\alpha B_\mu=\frac{\partial B_\mu}{\partial \mathrm{x}^\alpha}-\Gamma_{\alpha\mu}^\sigma B_\sigma$$

- The second term corrects for the change of the basis vectors in a curved space.
- This definition ensures that the derivative transforms as a tensor.

2.5.3 Relationship with Tensor

In this chapter, we investigate how a **tensor**, constructed from the derivative of a covariant vector V_m , behaves under a coordinate transformation. We show that the **ordinary derivative** of a vector does not yield a tensor, and that the **covariant derivative** is necessary to preserve a tensorial relation.

2.5.3.1 Transformation of a derivative

Consider the following definition of a rank 2 tensor in the *x* coordinate system:

$$T_{mn}(x) = \frac{\partial V_m(x)}{\partial x^n} \tag{33}$$

In another coordinate system *y*, we write:

$$T_{mn}(y) = \frac{\partial V_m(y)}{\partial y^n} \tag{34}$$

We now investigate whether $T_{mn}(x)$ actually behaves as a tensor, i.e., whether equation (34) corresponds to the transformed form of (33).

2.5.3.2 Expected tensor transformation

The usual **transformation formula for a covariant tensor** is:

$$T_{mn}(y) = \frac{\partial x^r}{\partial y^m}\frac{\partial x^s}{\partial y^n} T_{rs}(x) \tag{35}$$

Now substitute $T_{rs}(x) = \frac{\partial V_r(x)}{\partial x^s}$:

$$T_{mn}(y) = \frac{\partial x^r}{\partial y^m}\frac{\partial x^s}{\partial y^n}\frac{\partial V_r(x)}{\partial x^s} = \frac{\partial x^r}{\partial y^m}\frac{\partial V_r(x)}{\partial y^n} \tag{36}$$

Note that:

$$\frac{\partial V_r(x)}{\partial x^s} = \frac{\partial V_r(x)}{\partial y^n} \cdot \frac{\partial y^n}{\partial x^s} \text{ (via the chain rule)}$$

But the equation simplifies by considering directly:

$$T_{mn}(y) = \boxed{\frac{\partial x^r}{\partial y^m}\frac{\partial V_r(x)}{\partial y^n}} \tag{37}$$

We now want to show that:

$$\frac{\partial V_m(y)}{\partial y^n} \neq T_{mn}(y) \qquad ?$$

2.5.3.3 Calculation of $\frac{\partial V_m(y)}{\partial y^n}$

Use the transformation of vector components:

$$\mathrm{V_m}(\mathrm{y}) = \frac{\partial \mathrm{x^r}}{\partial \mathrm{y^m}} \mathrm{V_r}(\mathrm{x})$$

Then:

$$\frac{\partial V_m(y)}{\partial y^n} = \frac{\partial}{\partial y^n}\left(\frac{\partial x^r}{\partial y^m} V_r(x)\right)$$

Apply the product rule:

$$\frac{\partial V_m(y)}{\partial y^n} = \boxed{\frac{\partial x^r}{\partial y^m} \cdot \frac{\partial V_r(x)}{\partial y^n}} + \frac{\partial^2 x^r}{\partial y^n \partial y^m} \cdot V_r(x) \tag{38}$$

Then use the inverse transformation:

$$V_r(x) = \frac{\partial y^a}{\partial x^r} V_a(y) \tag{39}$$

Fill in (38):

$$\frac{\partial V_m(y)}{\partial y^n} = \boxed{\frac{\partial x^r}{\partial y^m} \cdot \frac{\partial V_r(x)}{\partial y^n}} + \frac{\partial y^a}{\partial x^r} \cdot \frac{\partial^2 x^r}{\partial y^n \partial y^m} \cdot V_a(y) \tag{40}$$

2.5.3.4 Connection with Christoffel symbols

Recall that (see earlier derivation of Christoffel symbol):

$$\Gamma^a_{nm} = \frac{\partial y^a}{\partial x^r} \cdot \frac{\partial^2 x^r}{\partial y^n \partial y^m} \tag{uit 6}$$

Substitution in (40) gives:

$$\frac{\partial V_m(y)}{\partial y^n} = \boxed{\frac{\partial x^r}{\partial y^m} \cdot \frac{\partial V_r(x)}{\partial y^n}} + \Gamma^a_{nm} V_a(y)$$

Reordering gives:

$$\boxed{\frac{\partial x^r}{\partial y^m} \frac{\partial V_r(x)}{\partial y^n}} = T_{mn}(y) = \frac{\partial V_m(y)}{\partial y^n} - \Gamma^{\mathrm{a}}_{\mathrm{nm}} V_a(y) \tag{41}$$

So:

$$T_{mn}(y) \neq \frac{\partial V_m(y)}{\partial y^n}$$

2.5.3.5 Covariant derivative of V_m

According to the above result:

$$T_{mn}(y) = \frac{\partial x^r}{\partial y^m}\frac{\partial V_r(x)}{\partial y^n} = \frac{\partial V_m(y)}{\partial y^n} - \Gamma^a_{nm} V_a(y)$$

And that is exactly the **covariant derivative of the covariant** vector V_m (see 2.5.2.7.4):

$$T_{mn}(y) = \frac{\partial V_m(y)}{\partial y^n} - \Gamma^a_{nm} V_a(y) = \nabla_n V_m(y)$$

$$\boldsymbol{T_{mn}(y) = \nabla_n V_m(y)} \tag{42}$$

2.5.3.6 Conclusion

- The ordinary derivative $\frac{\partial \mathrm{V_m(x)}}{\partial \mathrm{x^n}}$ is **not a tensor**.
- Only after correction with the **Christoffel symbol** does a quantity arise that behaves as a tensor under coordinate transformations.
- The correct tensorial version is the **covariant derivative**:

$$\boldsymbol{T_{mn} = \nabla_n V_m}$$

2.5.3.7 Covariant Differentiation for a Covariant Tensor

We now extend the concept **of covariant derivative** to a **covariant tensor of rank 2**. The derivation follows directly from the product rule for tensors and the previously established rules for the covariant derivatives of vectors.

2.5.3.7.1 Starting point

Consider a tensor $T_{\mu\nu}$, constructed as the product of two **covariant** vectors A_μ , and B_ν :

$$T_{\mu\nu} = A_\mu B_\nu$$

We now take the **covariant derivative** of this tensor with respect to x^α :

$$\nabla_\alpha T_{\mu\nu} = \nabla_\alpha (A_\mu B_\nu)$$

According to the product rule:

$$\nabla_\alpha T_{\mu\nu} = (\nabla_\alpha A_\mu) B_\nu + A_\mu (\nabla_\alpha B_\nu) \tag{a}$$

Now use the definition of the covariant derivative of a covariant vector (see section 2.5.2.2):

$$\nabla_\alpha A_\mu = \frac{\partial A_\mu}{\partial x^\alpha} - \Gamma^\beta_{\alpha\mu} A_\beta$$

$$\nabla_\alpha B_\nu = \frac{\partial B_\nu}{\partial x^\alpha} - \Gamma^\gamma_{\alpha\nu} B_\gamma$$

Substitute this into (a):

$$\nabla_\alpha T_{\mu\nu} = B_\nu \left\{ \frac{\partial A_\mu}{\partial \mathrm{x}^\alpha} - A_\beta \Gamma^\beta_{\alpha\mu} \right\} + A_\mu \left\{ \frac{\partial B_\nu}{\partial \mathrm{x}^\alpha} - B_\gamma \Gamma^\gamma_{\alpha\nu} \right\}$$

Work this out further:

$$\nabla_\alpha T_{\mu\nu} = B_\nu \frac{\partial A_\mu}{\partial \mathrm{x}^\alpha} - A_\beta B_\nu \Gamma^\beta_{\alpha\mu} + A_\mu \frac{\partial B_\nu}{\partial \mathrm{x}^\alpha} - A_\mu B_\gamma \Gamma^\gamma_{\alpha\nu}$$

$$\nabla_\alpha T_{\mu\nu} = B_\nu \frac{\partial A_\mu}{\partial \mathrm{x}^\alpha} + A_\mu \frac{\partial B_\nu}{\partial \mathrm{x}^\alpha} - A_\beta B_\nu \Gamma^\beta_{\alpha\mu} - A_\mu B_\gamma \Gamma^\gamma_{\alpha\nu}$$

$$\nabla_\alpha T_{\mu\nu} = \frac{\partial \left(A_\mu B_\nu \right)}{\partial \mathrm{x}^\alpha} - A_\beta B_\nu \Gamma^\beta_{\alpha\mu} - A_\mu B_\gamma \Gamma^\gamma_{\alpha\nu}$$

2.5.3.7.2 Final formula

Since $\boldsymbol{T_{\mu\nu}} = A_\mu B_\nu$, we ultimately obtain:

$$\boldsymbol{\nabla_\alpha T_{\mu\nu} = \frac{\partial T_{\mu\nu}}{\partial x^\alpha} - T_{\beta\nu}\Gamma^\beta_{\alpha\mu} - T_{\mu\gamma}\Gamma^\gamma_{\alpha\nu}} \tag{43}$$

2.5.3.7.3 Summary

The **covariant derivative of a covariant tensor $\boldsymbol{T_{\mu\nu}}$** consists of:

- the ordinary derivative $\boldsymbol{\frac{\partial T_{\mu\nu}}{\partial x^\alpha}}$,
- and two **correction terms** with Christoffel symbols, one for each index of the tensor.

This guarantees that $\nabla_\alpha T_{\mu\nu}$ behaves like a tensor under coordinate transformations.

2.5.3.8 Covariant Differentiation for a Contravariant Tensor

We now extend the concept of **covariant differentiation** further to a **contravariant tensor of rank 2**. This tensor has two upper indices and transforms differently than a covariant tensor. We again follow the product rule and apply the familiar covariant derivative formulas.

2.5.3.8.1 Starting point

Consider a contravariant tensor $T^{\mu\nu}$ as the product of two contravariant vectors:

$$T^{\mu\nu} = A^\mu B^\nu$$

The covariant derivative of $T^{\mu\nu}$ with respect to x^α is then:

Now differentiate this tensor covariantly:

$$\nabla_\alpha T^{\mu\nu} = B^\nu \nabla_\alpha A^\mu + A^\mu \nabla_\alpha B^\nu \tag{a}$$

Now use the formula for the covariant derivative of a contravariant vector (see section 2.5.2.7.4):

$$\nabla_{\alpha} A^{\mu} = \frac{\partial A^{\mu}}{\partial \mathrm{x}^{\alpha}} + \Gamma^{\mu}_{\beta\alpha}\, A^{\beta}$$

$$\nabla_{\alpha} B^{\nu} = \frac{\partial B^{\nu}}{\partial \mathrm{x}^{\alpha}} + \Gamma^{\nu}_{\gamma\alpha}\, B^{\gamma}$$

Substitution in (a) gives:

$$\nabla_{\alpha} T^{\mu\nu} = B^{\nu}\left\{\frac{\partial A^{\mu}}{\partial \mathrm{x}^{\alpha}} + A^{\beta}\,\Gamma^{\mu}_{\beta\alpha}\right\} + A^{\mu}\left\{\frac{\partial B^{\nu}}{\partial \mathrm{x}^{\alpha}} + B^{\gamma}\,\Gamma^{\nu}_{\gamma\alpha}\right\}$$

$$\nabla_{\alpha} T^{\mu\nu} = B^{\nu}\frac{\partial A^{\mu}}{\partial \mathrm{x}^{\alpha}} + A^{\beta} B^{\nu}\,\Gamma^{\mu}_{\beta\alpha} + A^{\mu}\frac{\partial B^{\nu}}{\partial \mathrm{x}^{\alpha}} + A^{\mu} B^{\gamma}\,\Gamma^{\nu}_{\gamma\alpha}$$

$$\nabla_{\alpha} T^{\mu\nu} = B^{\nu}\frac{\partial A^{\mu}}{\partial \mathrm{x}^{\alpha}} + A^{\mu}\frac{\partial B^{\nu}}{\partial \mathrm{x}^{\alpha}} + A^{\beta} B^{\nu}\,\Gamma^{\mu}_{\beta\alpha} + A^{\mu} B^{\gamma}\,\Gamma^{\nu}_{\gamma\alpha}$$

Rewrite this as:

$$\nabla_{\alpha} T^{\mu\nu} = \frac{\partial (A^{\mu} B^{\nu})}{\partial \mathrm{x}^{\alpha}} + A^{\beta} B^{\nu}\,\Gamma^{\mu}_{\beta\alpha} + A^{\mu} B^{\gamma}\,\Gamma^{\nu}_{\gamma\alpha}$$

2.5.3.8.2 Final formula

Since $T^{\mu\nu} = A^{\mu} B^{\nu}$, we get:

$$\boldsymbol{\nabla_{\alpha} T^{\mu\nu} = \frac{\partial T^{\mu\nu}}{\partial \mathrm{x}^{\alpha}} + T^{\beta\nu}\Gamma^{\mu}_{\beta\alpha} + T^{\mu\gamma}\Gamma^{\nu}_{\gamma\alpha}} \qquad (\mathbf{44})$$

2.5.3.8.3 In summary

The **covariant derivative of a contravariant tensor $\boldsymbol{T^{\mu\nu}}$** consists of:

- the ordinary derivative $\boldsymbol{\frac{\partial T^{\mu\nu}}{\partial \mathrm{x}^{\alpha}}}$,
- and two **correction terms** with Christoffel symbols, one for each upper index.

The order of indices in the Christoffel symbol is important: the **first index** (top) indicates which tensor index is adjusted, the two bottom ones come from the derivative.

2.5.3.9 Covariant Differentiation for a Mixed Tensor

We will now look at how the **covariant derivative** is applied to a **mixed** tensor—a tensor that has both a **contravariant** and a **covariant** index.

2.5.3.9.1 Starting point

Consider the mixed tensor T^{μ}_{ν} , defined as the product of a contravariant vector A^{μ} and a covariant vector B_{ν} :

$$T^{\mu}_{\nu} = A^{\mu} B_{\nu}$$

The covariant derivative of T^{μ}_{ν} with respect to x^{α} is:

$$\nabla_{\alpha} T^{\mu}_{\nu} = B_{\nu} \nabla_{\alpha} A^{\mu} + A^{\mu} \nabla_{\alpha} B_{\nu} \qquad (a)$$

2.5.3.9.2 Use of covariant derivatives

Replace the derivatives with their known expressions:

$$\nabla_\alpha A^\mu = \frac{\partial A^\mu}{\partial \mathrm{x}^\alpha} + \Gamma^\mu_{\beta\alpha}\, A^\beta$$

$$\nabla_\alpha B_\nu = \frac{\partial B_\nu}{\partial \mathrm{x}^\alpha} - \Gamma^\gamma_{\alpha\nu}\, B_\gamma$$

Substitute these into (a):

$$\nabla_\alpha T^\mu_\nu = B_\nu \left\{ \frac{\partial A^\mu}{\partial \mathrm{x}^\alpha} + A^\beta\, \Gamma^\mu_{\beta\alpha} \right\} + A^\mu \left\{ \frac{\partial B_\nu}{\partial \mathrm{x}^\alpha} - B_\gamma\, \Gamma^\gamma_{\alpha\nu} \right\}$$

$$\nabla_\alpha T^\mu_\nu = B_\nu \frac{\partial A^\mu}{\partial \mathrm{x}^\alpha} + A^\beta B_\nu \Gamma^\mu_{\beta\alpha} + A^\mu \frac{\partial B_\nu}{\partial \mathrm{x}^\alpha} - A^\mu B_\gamma \Gamma^\gamma_{\alpha\nu}$$

$$\nabla_\alpha T^\mu_\nu = B_\nu \frac{\partial A^\mu}{\partial \mathrm{x}^\alpha} + A^\mu \frac{\partial B_\nu}{\partial \mathrm{x}^\alpha} + A^\beta B_\nu \Gamma^\mu_{\beta\alpha} - A^\mu B_\gamma \Gamma^\gamma_{\alpha\nu}$$

Rewrite this as:

$$\nabla_\alpha T^\mu_\nu = \frac{\partial (A^\mu B_\nu)}{\partial \mathrm{x}^\alpha} + A^\beta B_\nu \Gamma^\mu_{\beta\alpha} - A^\mu B_\gamma \Gamma^\gamma_{\alpha\nu}$$

2.5.3.9.3 Final formula

Since $T^\mu_\nu = A^\mu B_\nu$, it follows that:

$$\boldsymbol{\nabla_\alpha T^\mu_\nu = \frac{\partial T^\mu_\nu}{\partial \mathrm{x}^\alpha} + T^\beta_\nu \Gamma^\mu_{\beta\alpha} - T^\mu_\gamma \Gamma^\gamma_{\nu\alpha}} \qquad (45)$$

2.5.4 Key insights

- **Christoffel symbols** Γ^k_{ij} describe how the basis vectors change from point to point in curved space.
- They are **not tensors**, but derivatives of the metric that determine **how vectors 'rotate'** with space.
- In flat space they are zero, in curved space they are not.
- The **covariant derivative** ∇ corrects the ordinary derivative with these symbols, so that the result **remains tensorial**.
- The **Levi-Civita connection** is the unique connection that satisfies:
  - $\nabla g = 0$ (metric remains constant)
  - No torsion: $\Gamma^k_{ij} = \Gamma^k_{ji}$

The **covariant derivative of a mixed tensor** T^μ_ν contains:

- the ordinary derivative $\boldsymbol{\frac{\partial T^\mu_\nu}{\partial x^\alpha}}$**,**

- a **positive correction term** for the contravariant index (μ),
- a **negative correction term** for the covariant index (ν).

This structure ensures that $\nabla_\alpha T^\mu_\nu$ behaves like a tensor under coordinate transformations.

2.5.5 Intuitive explanation

Imagine walking on a sphere with an arrow in your hand (a vector). If you walk in a straight line and try to keep the arrow pointing in the same direction:

- On a flat surface, this works fine—the arrow remains constant.
- On a sphere, your arrow will notice that it **"rotates" relative to the surface**, even though you are holding it straight.

That effect is measured by **Christoffel symbols**.

In curved space, a vector is no longer automatically 'equal' to its neighbors. The **covariant derivative** tells us how a vector **changes in a way that makes sense in curved geometry**.

Think of a compass trying to maintain a fixed direction while you walk on sloping terrain. It has to compensate for the slope and curvature of the terrain—which is exactly what the Christoffel symbol does.

Summary overview:

Concept	Meaning
Γ^k_{ij}	Compensation term in differentiation in curved space
Covariant derivative	Derivative that is "coordinate-free" and tensorial
$\nabla g V^i$	Ordinary derivative + correction via Γ^i_{jk}
Geometric meaning	Parallel transport, curvature, and change of direction in curved space

2.6 Geodesic equation and Christoffel symbols

As discussed earlier, Einstein attempted to formulate the **geometry of space-time** in such a way that a freely falling object does not experience gravity, but instead follows a "straight line" in curved space-time. Such a path is called a **geodesic**.

In this context, the **acceleration of the** object's **four-position** is equal to zero. In local free fall, the object therefore follows:

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0 \text{ with } ds = cd\tau$$

Here, τ is the proper time, measured by an observer in a freely falling coordinate system. The origin of this system "surrenders" to gravity and follows exactly the path of the free-falling object. A geodesic line is the shortest route (in proper time) between two points, given a certain space-time metric.

2.6.1 Explanation of the Terms

2.6.1.1 Local (freely falling) frame (ξ^α):

This is a coordinate system defined locally in spacetime.
It is called "freely falling" because the axes of this frame behave like a particle in free fall, meaning that at that moment no non-gravitational forces act on it.
On a very small scale (and as an approximation), the laws of physics in this system can be simplified, similar to the local laws in an inertial frame (moving in a straight line at constant speed).

2.6.1.2 General curved coordinate system (x^μ):

This is a global coordinate system that describes the entire spacetime, which is generally curved due to mass and energy.
The coordinates x^μ can be arbitrary coordinates used to specify points in a curved spacetime, without restriction to a local inertial frame.

2.6.1.3 The relation between the two

The theorem states that there exists a local transformation between these two frames, similar to a Lorentz transformation, that defines the relation between the local freely falling coordinates ξ^α and the general coordinates x^μ.

2.6.1.4 Meaning in physics

In general relativity, this concept expresses that in a curved spacetime one can always define a local "flat" coordinate system at any point.
In this local, freely falling frame, the laws of physics appear to operate in the same way as in a special relativistic inertial frame, which simplifies the local physics.
This is crucial for understanding the local effects of gravity: gravity is the manifestation of the curvature of spacetime itself, and in a local freely falling frame this curvature can be ignored.

2.6.2 Derivation via coordinate transformation

Suppose that ξ^α are the coordinates in the local (free-falling) system, while x^μ are the coordinates in a generally curved coordinate system. Then the following applies:

$$\xi^\alpha = \frac{\partial \xi^\alpha}{\partial x^\mu} x^\mu$$

The first derivative becomes:

$$\frac{d\xi^\alpha}{d\tau} = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}$$

The second derivative:

$$\frac{d^2\xi^\alpha}{d\tau^2} = \frac{d}{d\tau}\left(\frac{\partial \xi^\alpha}{\partial x^\mu}\cdot\frac{dx^\mu}{d\tau}\right) = \frac{d}{d\tau}\left(\frac{d\xi^\alpha}{dx^\mu}\right)\cdot\frac{dx^\mu}{d\tau} + \frac{\partial \xi^\alpha}{\partial x^\mu}\cdot\frac{d^2x^\mu}{d\tau^2}$$

$$\frac{d^2\xi^\alpha}{d\tau^2} = \frac{d}{d\tau}\left(\frac{d\xi^\alpha}{dx^\mu}\right)\cdot\frac{dx^\mu}{d\tau} + \frac{\partial \xi^\alpha}{\partial x^\mu}\cdot\frac{d^2x^\mu}{d\tau^2}$$

$$\frac{d^2\xi^\alpha}{d\tau^2} = \frac{d^2\xi^\alpha}{dx^\mu dx^\nu}\cdot\frac{dx^\nu}{d\tau}\cdot\frac{dx^\mu}{d\tau} + \frac{\partial \xi^\alpha}{\partial x^\mu}\cdot\frac{d^2x^\mu}{d\tau^2}$$

Because $\frac{d^2\xi^\alpha}{d\tau^2} = 0$ for a freely falling object, the following applies:

$$0 = \frac{d^2\xi^\alpha}{dx^\mu dx^\nu}\cdot\frac{dx^\nu}{d\tau}\cdot\frac{dx^\mu}{d\tau} + \frac{\partial \xi^\alpha}{\partial x^\mu}\cdot\frac{d^2x^\mu}{d\tau^2}$$

To return to the x-coordinates, we multiply both sides by $\frac{\partial x^\beta}{\partial \xi^\alpha}$:

$$0 = \frac{dx^\beta}{d\xi^\alpha}\cdot\frac{d^2\xi^\alpha}{dx^\mu dx^\nu}\cdot\frac{dx^\nu}{d\tau}\cdot\frac{dx^\mu}{d\tau} + \frac{\partial \xi^\alpha}{\partial x^\mu}\cdot\frac{d^2x^\mu}{d\tau^2}\cdot\frac{dx^\beta}{d\xi^\alpha}$$

Here, the following applies:

$$\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial x^\beta}{\partial \xi^\alpha} = \frac{\partial x^\beta}{\partial x^\mu} = \delta^\beta_\mu \quad (Kronecker\ delta)$$

So:

$$0 = \frac{\partial x^\beta}{\partial \xi^\alpha}\cdot\frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\nu}\cdot\frac{dx^\mu}{d\tau}\cdot\frac{\partial x^\nu}{\partial \tau} + \delta^\beta_\mu\cdot\frac{d^2x^\mu}{d\tau^2}$$

The ***Kronecker delta*** is defined as 1 only when $\beta = \mu$, and 0 when $\beta \neq \mu$.
$\frac{\partial x^\beta}{\partial x^\mu} = \delta^\beta_\mu = 0$, because x^β and x^μ are perpendicular to each other in the case $\beta \neq \mu$. Due to Einstein notation, the rightmost term consists of four elements, so $\delta^\beta_\mu = 1$ when $\mu = \beta$, while the other three terms are zero. This means that we can replace the μ index in the last term with β .

So:

$$0 = \frac{\partial x^{\beta}}{\partial \xi^{\alpha}} \cdot \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}} \cdot \frac{dx^{\mu}}{d\tau} \cdot \frac{\partial x^{\nu}}{\partial \tau} + \frac{d^2 x^{\beta}}{d\tau^2}$$

Recognize the **Christoffel symbol** here:

$$\Gamma^{\beta}_{\mu\nu} = \frac{\partial x^{\beta}}{\partial \xi^{\alpha}} \cdot \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}}$$

This gives us the **geodesic equation**:

$$\boldsymbol{\frac{d^2 x^{\beta}}{d\tau^2} + \Gamma^{\beta}_{\mu\nu} \frac{\partial x^{\mu}}{\partial \tau} \frac{\partial x^{\nu}}{\partial \tau} = 0} \qquad (\mathbf{1})$$

2.6.3 Interpretation

The second derivative $\frac{d^2 x^{\beta}}{d\tau^2}$ is therefore compensated by the **Christoffel term**. When there is **no gravity** (i.e., flat space-time), all $\Gamma^{\beta}_{\mu\nu} = 0$, and the object follows a straight line:

$$\frac{d^2 x^{\beta}}{d\tau^2} = 0$$

The geodesic equation describes the path of a freely falling particle in curved space-time, i.e., the path with the shortest distance in 4D space-time.

2.6.4 Summary

The relationship between acceleration in the local free-fall system and in the general coordinate system is:

$$\frac{d^2 \xi^{\beta}}{d\tau^2} = \frac{d^2 x^{\beta}}{d\tau^2} + \Gamma^{\beta}_{\mu\nu} \frac{\partial x^{\mu}}{\partial \tau} \frac{\partial x^{\nu}}{\partial \tau}$$

For an object on a **geodesic path,** the acceleration in the local system is zero:

$$0 = \frac{d^2 x^{\beta}}{d\tau^2} + \Gamma^{\beta}_{\mu\nu} \frac{\partial x^{\mu}}{\partial \tau} \frac{\partial x^{\nu}}{\partial \tau}$$

Or, written differently:

$$\frac{d^2 x^{\beta}}{d\tau^2} = -\Gamma^{\beta}_{\mu\nu} \frac{\partial x^{\mu}}{\partial \tau} \frac{\partial x^{\nu}}{\partial \tau}$$

2.6.5 Christoffel symbol formula

Where the Christoffel symbol contains the relationship between the moving system ξ^{α} and the "rest" system x^{β}.

$$\Gamma^{\beta}_{\mu\nu} = \frac{\partial x^{\beta}}{\partial \xi^{\alpha}} \cdot \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}}$$

2.6.6 Remarks

Note 1: Affine parameter

For massless particles such as photons, $\tau = 0$, which means that use of the eigentime is not suitable. We therefore use an **affine parameter** λ , so that the geodesic equation becomes:

$$0 = \frac{d^2 x^{\beta}}{d\lambda^2} + \Gamma^{\beta}_{\mu\nu} \frac{\partial x^{\mu}}{\partial \lambda} \frac{\partial x^{\nu}}{\partial \lambda}$$

The parameter λ often disappears in the final physical expressions, which facilitates its use.

Note 2: Speed of light *c*

In much of the literature, *c*= 1 is chosen for simplicity. In this document, however, we retain **the speed of light *c*** explicitly in the formulas. This makes it easier to check the **dimensions** and increases the transparency of the calculations.

2.6.7 Key insights

- **Geodesics** are the "straightest" possible lines in a curved space-time—think of the shortest route between two points on a sphere.
- In general relativity, geodesics describe the path that a **freely moving particle** follows under the influence of gravity (but without other forces).
- The **geodesic equation** is:

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{\nu\rho} \frac{dx^{\nu}}{d\tau} \frac{dx^{\rho}}{d\tau} = 0$$

- This is a second-order differential equation that determines the trajectory in terms of the Christoffel symbols $\Gamma^{\mu}_{\nu\rho}$.
- The equation shows that the curvature of space-time (via Γ) **determines** the **acceleration of the path**, without any external force.

2.6.8 Intuitive explanation

Imagine rolling an arrow across a sphere without touching it:

- The arrow follows the "straightest" line on the sphere—which is not a straight line in the usual sense, but a **large circle** such as the equator or a meridian.
- We call that path a **geodesic**.

In relativity theory:

- When you drop an apple, it does not follow a curve due to a force, but a geodesic in a curved space-time - **the curvature of the Earth determines the trajectory**.
- The **Christoffel symbols** in the equation describe how the path "deviates from straight ahead" depending on the geometry.

Think of a GPS that adjusts its own course depending on the curves in the landscape. That "correction" is the role of $\Gamma^{\mu}_{\nu\rho}$.

Table overview:

Magnitude	**Meaning**
$x^{\mu}(\tau)$	Coordinates of the particle as a function of proper time
$\frac{d^2x^{\mu}}{d\tau^2}$	Acceleration along the world line
$\Gamma^{\mu}_{\nu\rho}$	"Deflection coefficient" due to space-time curvature
Geodesic equation	Path without external forces: pure gravity

2.7 Christoffel symbols expressed in terms of the metric tensor

As discussed earlier, the **metric tensor** $g_{\mu\nu}$ contains all information about the curvature and geometry of space-time. In this chapter, we will show how the **Christoffel symbol** $\Gamma^{\beta}_{\mu\nu}$ can be expressed **exclusively in terms of the metric tensor and its derivatives**.

2.7.1 Conditions and definitions

We start from the following standard forms:

- **Metric tensor (from local flat space):**

$$g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\nu}}$$

where $\eta_{\alpha\beta} = diag(1, -1, -1, -1)$ is the Minkowski metric (see also section 5.6.1).

- Christoffel symbol (via transformation):

$$\Gamma^{\beta}_{\mu\nu} = \frac{\partial x^{\beta}}{\partial \xi^{\lambda}} \frac{\partial^2 \xi^{\lambda}}{\partial x^{\mu} \partial x^{\nu}}$$

2.7.2 Transformation via chain rule

We start by rewriting the metric tensor in a slightly different form $g_{\alpha\mu}$:

$$g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial\xi^\alpha}{\partial x^\mu} \frac{\partial\xi^\beta}{\partial x^\nu} \quad \text{because of symmetry} \Longrightarrow \quad g_{\nu\mu} = \eta_{\alpha\beta} \frac{\partial\xi^\alpha}{\partial x^\mu} \frac{\partial\xi^\beta}{\partial x^\nu}$$

By replacing the fictitious index α with σ :

$$\sigma \Longrightarrow \quad g_{\nu\mu} = \eta_{\sigma\beta} \frac{\partial\xi^\sigma}{\partial x^\mu} \frac{\partial\xi^\beta}{\partial x^\nu}$$

By replacing the index ν with α :

$$\alpha \Longrightarrow \quad g_{\alpha\mu} = \eta_{\sigma\beta} \frac{\partial\xi^\sigma}{\partial x^\mu} \frac{\partial\xi^\beta}{\partial x^\alpha} \tag{2}$$

Now we rewrite the Christoffel symbol by multiplying each part of the equation by the partial derivative of ξ^σ with respect to x^β :

$$\left(\frac{\partial\xi^\sigma}{\partial x^\beta}\right)\Gamma^\beta_{\mu\nu} = \frac{\partial x^\beta}{\partial\xi^\lambda} \frac{\partial^2\xi^\lambda}{\partial x^\mu \partial x^\nu} \left(\frac{\partial\xi^\sigma}{\partial x^\beta}\right) = \left(\frac{\partial x^\beta}{\partial\xi^\lambda} \frac{\partial\xi^\sigma}{\partial x^\beta}\right) \frac{\partial^2\xi^\lambda}{\partial x^\mu \partial x^\nu} \tag{3a}$$

Or:

$$\left(\frac{\partial x^\beta}{\partial\xi^\lambda} \frac{\partial\xi^\sigma}{\partial x^\beta}\right) = \frac{\partial\xi^\sigma}{\partial\xi^\lambda} = \delta^\sigma_\lambda \ or \ \delta^\sigma_\lambda \ \{= 1 \text{ if } \sigma = \lambda \text{ and } = 0 \text{ if } \ \sigma \neq \lambda\}$$

So together with (3a) this becomes:

$$\left(\frac{\partial\xi^\sigma}{\partial x^\beta}\right)\Gamma^\beta_{\mu\nu} = \delta^\sigma_\lambda \frac{\partial^2\xi^\lambda}{\partial x^\mu \partial x^\nu}$$

If $\sigma = \lambda$, then we replace λ door σ:

$$\left(\frac{\partial\xi^\sigma}{\partial x^\beta}\right)\Gamma^\beta_{\mu\nu} = \frac{\partial^2\xi^\sigma}{\partial x^\mu \partial x^\nu} \tag{3b}$$

So from (2):

$$\frac{\partial\, g_{\alpha\mu}}{\partial x^\nu} = \eta_{\sigma\beta} \frac{\partial^2\xi^\sigma}{\partial x^\nu \partial x^\mu} \frac{\partial\xi^\beta}{\partial x^\alpha} + \eta_{\sigma\beta} \frac{\partial\xi^\sigma}{\partial x^\mu} \frac{\partial^2\xi^\beta}{\partial x^\nu \partial x^\alpha}$$

With (3b) we can deduce:

$$\frac{\partial^2\xi^\sigma}{\partial x^\nu \partial x^\mu} = \frac{\partial\xi^\sigma}{\partial x^\rho}\Gamma^\rho_{\mu\nu} \ \text{and} \ \frac{\partial^2\xi^\beta}{\partial x^\nu \partial x^\alpha} = \frac{\partial\xi^\beta}{\partial x^\rho}\Gamma^\rho_{\nu\alpha}$$

Now we rewrite the partial derivative $g_{\alpha\mu}$ in relation with x^ν as follows:

$$\frac{\partial\, g_{\alpha\mu}}{\partial x^\nu} = \eta_{\sigma\beta} \frac{\partial\xi^\beta}{\partial x^\alpha} \frac{\partial\xi^\sigma}{\partial x^\rho}\Gamma^\rho_{\mu\nu} + \eta_{\sigma\beta} \frac{\partial\xi^\sigma}{\partial x^\mu} \frac{\partial\xi^\beta}{\partial x^\rho}\Gamma^\rho_{\nu\alpha}$$

We know from above:

$$\text{metric tensor:} \ g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial\xi^\alpha}{\partial x^\mu} \frac{\partial\xi^\beta}{\partial x^\nu}$$

So:

$$\eta_{\sigma\beta}\frac{\partial\xi^{\beta}}{\partial x^{\alpha}}\frac{\partial\xi^{\sigma}}{\partial x^{\rho}} = g_{\rho\alpha} \text{ and } \eta_{\sigma\beta}\frac{\partial\xi^{\sigma}}{\partial x^{\mu}}\frac{\partial\xi^{\beta}}{\partial x^{\rho}} = g_{\mu\rho}$$

$$\frac{\partial\, g_{\alpha\mu}}{\partial x^{\nu}} = g_{\rho\alpha}\,\Gamma^{\rho}_{\mu\nu} + g_{\mu\rho}\,\Gamma^{\rho}_{\nu\alpha} \tag{3c}$$

Perform cyclic permutations:

$$\frac{\partial\, g_{\alpha\nu}}{\partial x^{\mu}} = g_{\rho\alpha}\,\Gamma^{\rho}_{\nu\mu} + g_{\nu\rho}\,\Gamma^{\rho}_{\mu\alpha} \qquad \mu \text{ and } \nu \text{ are swapped} \tag{3d}$$

$$\frac{\partial\, g_{\mu\nu}}{\partial x^{\alpha}} = g_{\rho\mu}\,\Gamma^{\rho}_{\nu\alpha} + g_{\nu\rho}\,\Gamma^{\rho}_{\alpha\mu} \qquad \alpha \text{ and } \mu \text{ are swapped} \tag{3e}$$

Now take out (3c)+(3d)-(3e):

$$\frac{\partial\, g_{\alpha\mu}}{\partial x^{\nu}} + \frac{\partial\, g_{\alpha\nu}}{\partial x^{\mu}} - \frac{\partial\, g_{\mu\nu}}{\partial x^{\alpha}} = g_{\rho\alpha}\,\Gamma^{\rho}_{\mu\nu} + g_{\mu\rho}\,\Gamma^{\rho}_{\nu\alpha} + g_{\rho\alpha}\,\Gamma^{\rho}_{\nu\mu} + g_{\nu\rho}\,\Gamma^{\rho}_{\mu\alpha} - g_{\rho\mu}\,\Gamma^{\rho}_{\nu\alpha} - g_{\nu\rho}\,\Gamma^{\rho}_{\alpha\mu}$$

Due to symmetry in the indices, this simplifies to:

$$\frac{\partial\, g_{\alpha\mu}}{\partial x^{\nu}} + \frac{\partial\, g_{\alpha\nu}}{\partial x^{\mu}} - \frac{\partial\, g_{\mu\nu}}{\partial x^{\alpha}} = 2g_{\rho\alpha}\,\Gamma^{\rho}_{\mu\nu}$$

$$g_{\rho\alpha}\,\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\left(\frac{\partial\, g_{\alpha\mu}}{\partial x^{\nu}} + \frac{\partial\, g_{\alpha\nu}}{\partial x^{\mu}} - \frac{\partial\, g_{\mu\nu}}{\partial x^{\alpha}}\right)$$

2.7.3 Isolating the Christoffel symbol

The final step is to multiply both sides of the equation by the inverse metric tensor $g^{\rho\alpha}$ to find the Christoffel symbol:

$$g^{\rho\alpha}\,g_{\rho\alpha}\,\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\,g^{\rho\alpha}\left(\frac{\partial\, g_{\alpha\mu}}{\partial x^{\nu}} + \frac{\partial\, g_{\alpha\nu}}{\partial x^{\mu}} - \frac{\partial\, g_{\mu\nu}}{\partial x^{\alpha}}\right)$$

Swapping ρ to β:

$$\Gamma^{\beta}_{\mu\nu} = \frac{1}{2}\,g^{\beta\alpha}\left(\frac{\partial\, g_{\alpha\mu}}{\partial x^{\nu}} + \frac{\partial\, g_{\alpha\nu}}{\partial x^{\mu}} - \frac{\partial\, g_{\mu\nu}}{\partial x^{\alpha}}\right)$$

Usually, the following convention is adopted for writing partial derivatives:

$$\frac{\partial\, g_{\alpha\mu}}{\partial x^{\nu}} \equiv g_{\alpha\mu\,,\nu}$$

So, the **Christoffel symbol** in compact notation:

$$\Gamma^{\beta}_{\mu\nu} = \frac{1}{2}\,g^{\beta\alpha}\left(g_{\alpha\mu\,,\nu} + g_{\alpha\nu\,,\mu} - g_{\mu\nu\,,\alpha}\right)$$

2.7.4 Summary

The **Christoffel symbols** are fully expressed in terms of the **metric tensor** $\boldsymbol{g_{\mu\nu}}$ and its **first derivatives**:

$$\boxed{\Gamma^{\beta}_{\mu\nu} = \frac{1}{2}\, g^{\beta\alpha}\left(\frac{\partial\, g_{\alpha\mu}}{\partial x^{\nu}} + \frac{\partial\, g_{\alpha\nu}}{\partial x^{\mu}} - \frac{\partial\, g_{\mu\nu}}{\partial x^{\alpha}}\right)}$$

Or in short notation:

$$\boxed{\Gamma^{\beta}_{\mu\nu} = \frac{1}{2}\, g^{\beta\alpha}\left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}\right)}$$

2.7.5 Note

This equation is fundamental in general relativity. It shows that **the geometry of space-time**, and thus gravity, is completely determined by the metric. The Christoffel symbols describe how vectors change under parallel transport and appear in the geodesic equation, the covariant derivative, and later in the Riemann and Ricci tensors.

2.7.6 Key insights

- **Christoffel symbols** $\Gamma^{\lambda}_{\mu\nu}$ can be calculated entirely on the basis of the **metric tensor** $g_{\mu\nu}$.
- The explicit formula is:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\mathrm{g}^{\lambda\rho}\left(\partial_{\mu} g_{\rho\nu} + \partial_{\nu} g_{\rho\mu} - \partial_{\rho} g_{\mu\nu}\right)$$

- The Christoffel symbols indicate how coordinate systems are locally curved—and thus how vectors and trajectories behave.
- The **symmetry** $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$ is preserved as long as the metric is symmetric (which is always the case).
- This relationship forms the **bridge between geometry and dynamics** in general relativity.

2.7.7 Intuitive explanation

The **metric tensor** $g_{\mu\nu}$ tells you how to measure distances in a space (e.g., how "far" something is in curved coordinates).

But: if you are moving in a landscape and you want to know **how the direction of an arrow changes as you move forward**, you need more than just distances: you need to know **how the measuring sticks themselves change**. That is exactly what the **Christoffel symbols** do.

You can think of it this way:

- **Metrics** tell you *what is straight at a point*.
- The **Christoffel symbols** tell you *how "straight" changes as you move*.

You don't need to measure the change in basis vectors separately—you can **calculate it entirely from the metric itself**!

Table overview:

Quantity	Meaning
$g_{\mu\nu}$	Determines local distance and angle
$\partial_\sigma g_{\mu\nu}$	How the distance definition changes when you move
$\Gamma^\lambda_{\mu\nu}$	How base vectors change - determines deviation from "straight ahead"
Formula	Derivatives of metric converted with inverse metric

2.8 Geodesic Equation and its Newtonian Limit

Newtonian gravity describes how matter generates a gravitational potential Φ , and how, according to Newton's second law, that potential leads to acceleration:

$$\vec{a} = -\nabla\Phi$$

Here, Φ is the gravitational potential, and ∇ is the Euclidean gradient operator

$$\left(\frac{\partial}{\partial x}\hat{e}_x + \frac{\partial}{\partial y}\hat{e}_y + \frac{\partial}{\partial z}\hat{e}_z\right)$$

Here,$\hat{e}_x, \hat{e}_y, \hat{e}_z$ are the unit vectors along the respective axes. This description is accurate at low speeds, weak fields, and in a static regime. We will now show that the geodesic equation of general relativity reduces to the Newtonian gravitational equation in this limit.

2.8.1 Assumptions for the Newtonian limit:

- The particle moves slowly compared to the speed of light.
- The gravitational field is weak.
- The field is static, so it does not change with time.

2.8.2 Starting point: the geodesic equation

The geodesic equation describes the worldline of a particle that is only influenced by gravity. We will now show that in the context of the Newtonian limit, the geodesic equation reduces to Newton's gravitational equation.

From the previous chapter, we know that the geodesic equations, with proper time as a parameter of the worldline, are as follows:

$$\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta_{\mu\nu}\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\nu}{\partial\tau} = 0$$

The second term includes a sum over μ and ν over all indices, which amounts to 16 terms. Because the particle moves very slowly relative to the speed of light, the time component, i.e., the 0^e component of the particle's vector, dominates the other spatial components. We then arrive at the following approximation:

$$\text{with } \frac{dx^i}{d\tau} \ll \frac{dt}{d\tau} \quad \text{(because we know that } c\partial t = \partial x^0\text{)}$$

$$\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta_{\mu\nu}\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\nu}{\partial\tau} = 0$$

The only term that remains after approximation is the time component, where Γ^i_{00} , and $\mu = \nu = 0$. This gives:

$$\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta_{00}\left(\frac{cdt}{d\tau}\right)^2 = 0$$

Greek letters are normally used for the indices when describing four-dimensional space-time, but when only three-dimensional space is considered, it is customary to use Latin letters. Therefore, β is replaced by *i* (*i*= *x*, *y*, *z*), resulting in:

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i_{00}\left(\frac{cdt}{d\tau}\right)^2 = 0 \qquad (1)$$

2.8.3 Approximation of the Christoffel symbol

From the chapter *Christoffel symbols expressed in terms of the metric tensor*(2.7), it appears that the Christoffel symbol can be calculated with respect to the components of a given metric (where $x^0 \equiv \tau$) :

$$\Gamma^i_{00} = \frac{1}{2}\, g^{ij}\left(\frac{\partial\, g_{j0}}{\partial x^0} + \frac{\partial\, g_{j0}}{\partial x^0} - \frac{\partial\, g_{00}}{\partial x^j}\right)$$

Since the field is static, according to the second assumption of the Newtonian limit, the time derivative is $\frac{\partial\, g_{j0}}{\partial x^0} = 0$, so that the Christoffel symbol can be simplified to:

$$\Gamma^i_{00} = -\frac{1}{2}\, g^{ij}\,\frac{\partial\, g_{00}}{\partial x^j} \qquad (2)$$

2.8.4 Weak field approximation

If the gravitational field is weak enough, space-time will only be slightly distorted relative to the gravity-free Minkowski space-time of Special Relativity. Then the space-time metric can be considered a small perturbation of the Minkowski metric $\eta_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with } \left|h_{\mu\nu}\right| \ll 1$$

$$\frac{d\, g_{00}}{dx^j} = \frac{d(\eta_{00} + h_{00})}{dx^j}$$

$$\frac{d\, g_{00}}{dx^j} = \frac{d\eta_{00}}{dx^j} + \frac{d\, h_{00}}{dx^j} = 0 + \frac{d\, h_{00}}{dx^j} \quad \text{if } \eta_{00} = 1$$

For g_{00} , the following applies:

$$\frac{d\, g_{00}}{dx^j} = \frac{dh_{00}}{dx^j} \qquad (3)$$

So from (2) and (3) we get equation (1):

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{00}\left(\frac{cdt}{d\tau}\right)^2$$

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\, g^{ij}\,\frac{\partial\, h_{00}}{\partial x^j}\left(\frac{cdt}{d\tau}\right)^2$$

By defining $g^{ij} = \eta^{ij} - h^{ij}$, we find that $g^{\mu\sigma}\, g_{\sigma\nu} = \delta^\mu_\nu$, which corresponds to the first order of $h_{ij\,,}$ when defining an inverse metric.

We then obtain:

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\,\eta^{ij}\,\frac{\partial\, h_{00}}{\partial x^j}\left(\frac{cdt}{d\tau}\right)^2$$

But since η^{ij} is not zero for *j*=*i*, then $\eta^{ii} = -1$ (where *i* refers to the spatial components) applies:

$$\frac{d^2x^i}{d\tau^2} = -\frac{1}{2}\frac{\partial\, h_{00}}{\partial x^i}\left(\frac{cdt}{d\tau}\right)^2$$

We will now change the derivative on the left-hand side of τ naar t, which is done as follows:
First, in the above equation, *i* is replaced by 0, so that $x^0 = t$:

$$c^2\,\frac{d^2t}{d\tau^2} = -\frac{1}{2}\frac{\partial\, h_{00}}{\partial t}\left(\frac{cdt}{d\tau}\right)^2$$

Since the gravitational field is constant, the following applies: $\frac{\partial\, h_{00}}{\partial t} = 0$:

$$c^2\,\frac{d^2t}{d\tau^2} = 0 \;\Longrightarrow\; \frac{d^2t}{d\tau^2} = 0 \tag{4}$$

2.8.5 Switching to coordinate time

Next, we manipulate the partial derivatives with respect to tau (τ) :

$$\frac{d^2x^i}{d\tau^2} = \frac{d}{d\tau}\frac{dx^i}{d\tau} = \frac{d}{d\tau}\left(\frac{dt}{d\tau}\frac{dx^i}{dt}\right)$$

$$= \frac{dt}{d\tau}\left(\frac{d}{d\tau}\frac{dx^i}{dt}\right) + \frac{dx^i}{dt}\left(\frac{d}{d\tau}\frac{dt}{d\tau}\right)$$

$$= \frac{dt}{d\tau}\left(\frac{dt}{d\tau}\frac{d}{dt}\frac{dx^i}{dt}\right) + \frac{dx^i}{dt}\left(\frac{d}{d\tau}\frac{dt}{d\tau}\right)$$

$$= \left(\frac{dt}{d\tau}\right)^2\left(\frac{d^2x^i}{dt^2}\right) + \frac{dx^i}{dt}\left(\frac{d^2t}{d\tau^2}\right)$$

As we saw above in (4) $\frac{d^2t}{d\tau^2} = 0$:

$$\frac{d^2x^i}{d\tau^2} = \left(\frac{dt}{d\tau}\right)^2\left(\frac{d^2x^i}{dt^2}\right) = -\frac{1}{2}\frac{\partial\, h_{00}}{\partial x^i}\left(\frac{cdt}{d\tau}\right)^2 = -\frac{c^2}{2}\frac{\partial\, h_{00}}{\partial x^i}\left(\frac{dt}{d\tau}\right)^2$$

$$\Rightarrow \left(\frac{d^2x^i}{dt^2}\right)\left(\frac{\cancel{dt}}{d\tau}\right)^2 = -\frac{c^2}{2}\frac{\partial\, h_{00}}{\partial x^i}\left(\frac{\cancel{dt}}{d\tau}\right)^2$$

From this it follows that:

$$\frac{d^2x^i}{dt^2} = -\frac{c^2}{2}\frac{\partial\, h_{00}}{\partial x^i}$$

In general:

$$\frac{d^2\mathrm{x}}{dt^2}i + \frac{d^2\mathrm{y}}{dt^2}j + \frac{d^2\mathrm{z}}{dt^2}k = -\frac{\partial}{\partial \mathrm{x}}\left(\frac{c^2\, h_{00}}{2}\right)i - \frac{\partial}{\partial \mathrm{y}}\left(\frac{c^2\, h_{00}}{2}\right)j - \frac{\partial}{\partial \mathrm{z}}\left(\frac{c^2\, h_{00}}{2}\right)k$$

$$\frac{d^2\mathrm{x}}{dt^2}i + \frac{d^2\mathrm{y}}{dt^2}j + \frac{d^2\mathrm{z}}{dt^2}k = -\left[\frac{\partial}{\partial \mathrm{x}}i + \frac{\partial}{\partial \mathrm{y}}j + \frac{\partial}{\partial \mathrm{z}}k\right]\left(\frac{c^2\, h_{00}}{2}\right) = -\nabla\left(\frac{c^2\, h_{00}}{2}\right)$$

2.8.6 Equation in Newton

In vector form:

$$\boxed{\boldsymbol{\frac{d^2\vec{\mathrm{r}}}{dt^2} = -\nabla\phi \quad \mathbf{or} \quad \frac{d^2\vec{\mathrm{r}}}{dt^2} = -\overrightarrow{\mathbf{grad}}\,\phi}}$$

$$\mathbf{where}\ \boldsymbol{\phi = \frac{c^2h_{00}}{2}}\ \ \mathbf{and\ thus}\ \boldsymbol{h_{00} = \frac{2\phi}{c^2}}.$$

This is another way of writing Newton's law of gravitation

$$\vec{a} = -\nabla\Phi$$

.

2.8.7 Metric component in terms of potential

By writing the metric g_{00} as:

$$g_{00} = \eta_{00} + h_{00} = 1 + \frac{2\phi}{c^2} \tag{5}$$

the direct link between the metric tensor (component g_{00}) on the left and the gravitational potential ϕ on the right can be seen.

2.8.8 Example: calculation ofh_{00} on Earth

The value of h_{00} on Earth can now be calculated and checked to see if this value is negligible, meaning that the deviation from the Minkowski metric, due to the gravitational field, is negligible.

$$h_{00} = \frac{2\phi}{c^2}\,, \qquad \phi = \frac{GM_{earth}}{R_{earth}}$$

Or:

$$h_{00} = \frac{2GM}{c^2R}$$

With:

- $G = 6.67 \times 10^{-11}\ m^3.kg^{-1}.s^{-2}$
- $M_{earth} \simeq 6 \times 10^{24} kg \quad R_{earth} \simeq 6400\ km$
- $c \simeq 3 \times 10^8\ m.s^{-1}$

We obtain:

$$h_{00} \simeq \frac{2 \cdot 6.67 \cdot 10^{-11} \cdot 6 \cdot 10^{24}}{6.4 \cdot 10^6 \cdot (9 \cdot 10^{16})} \simeq 10^{-9}$$

For the Sun, this is $\sim 10^{-6}$ and for a white dwarf $\sim 10^{-4}$, which confirms that the weak field approximation is generally valid in many realistic situations.

2.8.9 Key insights

- In general relativity, free particles follow a **geodesic** in curved space-time.
- In the classical case (Newton), a particle follows a trajectory under the influence of the gravitational force:

$$\vec{a} = -\nabla\Phi$$

where Φ is the gravitational potential.

- In the **weak field approximation** and for **slow speeds,** the geodesic equation reduces to this Newtonian form.
- This requires that:
  - The space-time is barely curved $\Longrightarrow g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$
  - Only g_{00} deviates significantly from the flat Minkowski metric $\Longrightarrow g_{00} \approx -\left(1 + \frac{2\phi}{c^2}\right)$
- In this approach, the component Γ^i_{00} turns out to be equal to $\partial^i\phi$, which leads to Newton's gravitational equation.

2.8.10 Intuitive explanation

Einstein's theory must make the same predictions as Newton's theory in everyday life. That is to say:

- When gravity **is weak** (e.g., around the Earth),
- And the speeds **are much smaller than the speed of light** (e.g., falling apples),
- Then the relativistic formula must **transition** into the classical one.

The geodesic equation says: "a particle moves in curved space-time, without force."
But in weak fields, you can write that curvature as a small deviation from flat space. That deviation can then be seen as an "effective force" — exactly as Newton described it!

So: Newton's gravity is a **limiting case** of general relativity. The apple falls, not because of a force, but because the time component g_{00} is slightly curved by the mass of the Earth.

Summary comparison table:

Theory	Formula	Interpretation
Newton (classical)	$\vec{a} = -\nabla\Phi$	Acceleration due to force
Einstein (weak limit)	$\frac{d^2x^i}{dr^2} = -\Gamma^i_{00}$	Deviation from a straight line due to time curvature
Link between the two	$\Gamma^i_{00} = \frac{1}{2}\partial^i g_{00} \approx \partial^i \phi$	g_{00} encodes the potential

2.9 Generalizing the Definition of the Metric Tensor

In the previous sections, we saw how the geodesic equation is generalized from an inertial frame to an arbitrary coordinate system. In a similar way, we now extend the definition of the line element from flat Minkowski space-time to a general curved space-time — a so-called *pseudo-Riemannian manifold.* This structure forms the mathematical basis of general relativity.

2.9.1 The Minkowski line element in a local inertial frame

In a local inertial frame, we use the coordinates ξ^α , defined as:

$$\xi^0 = ct, \qquad \xi^1 = x, \qquad \xi^2 = y, \qquad \xi^3 = z$$

The Minkowski line element can be described as follows (see also Independence of the Chosen Coordinate System2.2.2 equation equation_2_2_4_2 and see also 5.6.1 Detailed Explanation of the Metric Tensor)

The corresponding line element is:

$$ds^2 = \eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta$$

where $\eta_{\alpha\beta}$ is the Minkowski metric:

$$\eta_{\alpha\beta} \equiv \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

2.9.2 Coordinate transformation to a general system

We now move to an arbitrary, possibly curved coordinate system x^μ , in which the old coordinates ξ^α are functions of the new ones:

$$\xi^\alpha = \xi^\alpha(x^0, x^1, x^2, x^3)$$

The differential change $d\xi^\alpha$ is then obtained via the chain rule:

$$d\xi^\alpha = \frac{\partial \xi^\alpha}{\partial x^0} dx^0 + \frac{\partial \xi^\alpha}{\partial x^1} dx^1 + \frac{\partial \xi^\alpha}{\partial x^2} dx^2 + \frac{\partial \xi^\alpha}{\partial x^3} dx^3$$

Using the Einstein summation convention:

$$d\xi^\alpha = \frac{\partial \xi^\alpha}{\partial x^\mu} dx^\mu \ \text{ en } \ d\xi^\beta = \frac{\partial \xi^\beta}{\partial x^\nu} dx^\nu$$

This allows us to rewrite the line element as:

$$ds^2 = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} dx^\mu dx^\nu$$

2.9.3 Definition of the general metric tensor

We now define the metric tensor $g_{\mu\nu}$ as:

$$\boldsymbol{metric\ tensor}\text{: } \boldsymbol{g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu}}$$

So that the line element in the new system becomes:

$$\boldsymbol{ds^2 = g_{\mu\nu} dx^\mu dx^\nu}$$

2.9.4 Properties of the metric tensor

The properties of the metric tensor are:

- **Symmetry:**

$$g_{\mu\nu} = g_{\nu\mu}$$

This follows directly from the definition, since the Minkowski metric is symmetric.

- **Inverse metric:**

$$g^{\mu\nu} \ \text{ zodanig dat } \ g^{\mu\nu} g_{\mu\nu} = \delta^\mu_\nu$$

where δ^μ_ν is the Kronecker delta.

- **Covariant versus contravariant**

The inverse $g^{\mu\nu}$ is called the **contravariant metric**; $g_{\mu\nu}$ is the **covariant metric.**

2.9.5 Importance of the metric in relativity

The metric tensor contains **all information about the structure of space-time**. It determines distances, angles, curvature, and thus also the behavior of objects under the influence of gravity. In the context of general relativity, gravity is nothing more than a manifestation of the curvature of space-time. This curvature is completely described by the metric.

Therefore, the **fundamental goal of general relativity** is to find $g_{\mu\nu}$ - the metric - as a solution to Einstein's field equations. Once known, this tensor determines the course of free motion, the curvature of space and time, and the interaction with energy and mass.

2.9.6 Number of independent components

Although the metric tensor $g_{\mu\nu}$ appears at first glance to contain 16 components (in a 4×4 matrix), it is symmetrical: $g_{\mu\nu} = g_{\nu\mu}$. Therefore only 10 independent components remain. These ten functions of space-time form the unknowns in Einstein's field equations.

2.9.7 Key insights

- The **metric tensor** $g_{\mu\nu}$ defines the distance in space-time via the line element:

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

- This formula applies in every coordinate system—flat or curved—as long as $g_{\mu\nu}$ is correctly adjusted.
- The metric is:
    - **Symmetrical:** $g_{\mu\nu} = g_{\nu\mu}$
    - **Tensor**: changes according to tensor transformations when coordinates are changed.
- The metric contains **all information about the local geometry**: distance, angle, volume, and light cones.
- In curved space, the metric is **location-dependent**: $g_{\mu\nu} = g_{\mu\nu}(x)$
- Through generalization, the metric becomes the **fundamental object** on which all other geometric quantities are based (Christoffel symbols, Riemann tensor, etc.).

2.9.8 Intuitive explanation

In special relativity, distance in space-time is something like:

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

That is the Minkowski metric: flat and constant

In general relativity, we say: **space-time itself is deformable**, so the distance formula must be **adapted to the curvature**.
We do this with a **metric tensor $g_{\mu\nu}$**, which tells us **how space and time are measured** at each point.

You can think of it as a measuring stick that **changes shape locally** depending on where you stand. Sometimes a "meter" is less or more than elsewhere, and angles can be skewed—depending on the mass/energy in the vicinity.

The generalization means that we no longer have a universal, fixed formula for distance, but a **flexible field** that is different at every point—and behaves **tensorially**.

Table overview:

Quantity	Meaning
$\boldsymbol{g_{\mu\nu}(x)}$	Local measurement recipe for space-time
Symmetry	$\boldsymbol{g_{\mu\nu} = g_{\nu\mu}}$
Tensor transformation	Metric adapts to coordinate change
Distance	$ds^2 = \boldsymbol{g_{\mu\nu}} dx^{\boldsymbol{\mu}} dx^{\boldsymbol{\nu}}$
Special limit	$\boldsymbol{g_{\mu\nu} = \eta_{\mu\nu}}$ (Minkowski metric)

2.10 The Riemann curvature tensor

The **Riemann curvature tensor** is one of the most important concepts in general relativity. This tensor describes how space-time is locally curved as a result of the presence of mass and energy. It determines how vectors change during parallel transport along curved paths around a closed loop.

In flat, Euclidean space, where there are no gravitational effects, the Riemann tensor disappears:

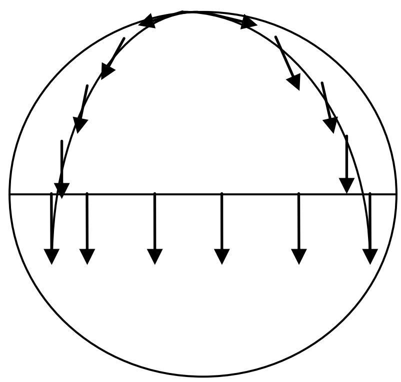
$$R^{\rho}_{\sigma\mu\nu} = 0\ (in\ flat\ space)$$

In this chapter, we derive the Riemann tensor in two ways:

1. Via the **commutator of two covariant derivatives**
2. Via the method of **geodesic deviation**

2.10.1 Derivation via the Commutator of Covariant Derivatives.

Using the concept of parallel transport of vectors or tensors, we will derive the expression for the Riemann tensor.



Parallel transport of a vector around a closed loop

An intuitive example of curvature can be found on the Earth's surface. Suppose we walk from the North Pole along a meridian to the equator with a stick held horizontally. There we turn 90 degrees, walk across the equator, and return to the North Pole via another meridian. Although we keep the stick in the "same direction," it points in a different direction after our return. This difference is due to the curvature of the surface.

In a similar way, we can transport a vector parallel in an infinitesimal loop on a manifold. In flat space, the vector does not change; in curved space, it does. This difference in parallel transport is directly linked to the Riemann tensor.

We define parallel transport as a movement in which the **covariant derivative** of a vector is zero. To derive the Riemann tensor, we examine how the result of twice covariant differentiation depends on the sequence of transporttion. The **commutator** of the covariant derivatives gives us that measure of curvature.

2.10.1.1Covariant Derivative Commutator

A commutator here refers to the difference between two operations, one performed in one direction and the other in the opposite direction. The commutator is defined as:

$$[AB] = AB - BA$$

The commutator is therefore only zero when the order of the two operations is irrelevant.

To obtain the Riemann tensor, the covariant derivative is chosen as the operation. The commutator of two covariant derivatives measures the difference between transporting the tensor first in one direction and then in the opposite direction. Thus, as a measure of the difference of the tensor along the path, the covariant derivative of the tensor is used.

In a flat space, the order of covariant derivatives makes no difference, because covariant differentiation falls back on partial differentiation, and therefore the commutator must yield zero. Conversely, any non-zero result of applying the commutator to covariant differentiation can be attributed to the curvature of space, and this is therefore referred to as the **Riemann tensor**.

2.10.1.2 Derivation of the Riemann Tensor.

The goal now is to derive the Riemann tensor by finding the following commutator:

$$[\nabla_{\mathrm{c}}, \nabla_{\mathrm{b}}]V_a = \nabla_{\mathrm{c}}\nabla_{\mathrm{b}}V_a - \nabla_{\mathrm{b}}\nabla_{\mathrm{c}}V_a$$

We know that the covariant derivative of V_a is given by (see equation 32):

$$\nabla_{\mathrm{b}}V_a = \frac{\partial V_a}{\partial x^b} - \Gamma^d_{\alpha b} V_d$$

And that this derivative itself is a tensor.

As we saw in the previous chapter:
(see equation 42)

$$\mathrm{T_{mn}}(\mathrm{y}) = \nabla_{\mathrm{n}}V_m = \frac{\partial \mathrm{V_m}}{\partial \mathrm{y^n}} - \Gamma^r_{nm}\mathrm{V_r}(\mathrm{x})$$

This means that:

$$T_{\mathrm{ab}}(\mathrm{y}) = \nabla_{\mathrm{b}}V_a = \frac{\partial \mathrm{V_a}}{\partial \mathrm{y^b}} - \Gamma^r_{ba}\mathrm{V_r}(\mathrm{x})$$

So, the covariant derivative of a vector ($\nabla_{\mathrm{b}}V_a$) is a tensor (see equation 42).

The covariant derivative of a tensor is (see equation 43):

$$\nabla_\alpha T_{\mu\nu} = \frac{\partial T_{\mu\nu}}{\partial \mathrm{x}^\alpha} - T_{\beta\nu}\Gamma^\beta_{\alpha\mu} - T_{\mu\gamma}\Gamma^\gamma_{\alpha\nu}$$

$$=> \nabla_{\mathrm{c}}T_{ab} = \frac{\partial T_{ab}}{\partial \mathrm{x^c}} - T_{eb}\Gamma^e_{ca} - T_{ae}\Gamma^e_{cb}$$

This results in:

$$\nabla_{\mathrm{c}}\nabla_{\mathrm{b}}V_a = \frac{\partial}{\partial x^c}(\nabla_{\mathrm{b}}V_a) - \Gamma^e_{\alpha c}\nabla_{\mathrm{b}}\mathrm{V_e} - \Gamma^e_{bc}\nabla_{\mathrm{e}}\mathrm{V_a} \tag{1}$$

The first term on the right-hand side:

$$\frac{\partial}{\partial x^c}(\nabla_{\mathrm{b}}V_a) = \frac{\partial^2 V_a}{\partial x^c \partial x^b} - \frac{\partial}{\partial x^c}\left(\Gamma^d_{\alpha b}V_d\right) \tag{1a}$$

$$\frac{\partial}{\partial x^c}(\nabla_{\mathrm{b}}V_a) = \frac{\partial^2 V_a}{\partial x^c \partial x^b} - \Gamma^d_{\alpha b}\frac{\partial V_d}{\partial x^c} - V_d\frac{\partial \Gamma^d_{\alpha b}}{\partial x^c} \tag{1b}$$

The second and third terms on the right-hand side:

$$\Gamma^e_{\alpha c}\nabla_{\mathrm{b}}\mathrm{V_e} = \Gamma^e_{\alpha c}\left(\frac{\partial V_e}{\partial x^b} - \Gamma^d_{be}V_d\right) \tag{1c}$$

$$\Gamma^e_{bc}\nabla_{\mathrm{e}}\mathrm{V_a} = \Gamma^e_{bc}\left(\frac{\partial V_a}{\partial x^e} - \Gamma^d_{ae}V_d\right) \tag{1d}$$

By combining the three terms (1b, 1c, 1d) in (1), we get:

$$\nabla_c \nabla_b V_a = \frac{\partial^2 V_a}{\partial x^c \partial x^b} - \Gamma^d_{ab}\frac{\partial V_d}{\partial x^c} - V_d \frac{\partial \Gamma^d_{ab}}{\partial x^c} - \Gamma^e_{ac}\left(\frac{\partial V_e}{\partial x^b} - \Gamma^d_{be} V_d\right) - \Gamma^e_{bc}\left(\frac{\partial V_a}{\partial x^e} - \Gamma^d_{ae} V_d\right) \qquad (1e)$$

By swapping *b* and *c*, we find:

$$\nabla_b \nabla_c V_a = \frac{\partial^2 V_a}{\partial x^b \partial x^c} - \Gamma^d_{ac}\frac{\partial V_d}{\partial x^b} - V_d \frac{\partial \Gamma^d_{ac}}{\partial x^b} - \Gamma^e_{ab}\left(\frac{\partial V_e}{\partial x^c} - \Gamma^d_{ce} V_d\right) - \Gamma^e_{cb}\left(\frac{\partial V_a}{\partial x^e} - \Gamma^d_{ae} V_d\right) \qquad (2)$$

By subtracting (1e)-(2), the first and last terms cancel each other out. Since the Christoffel symbol is symmetric with respect to the lower indices, we get:

$$\nabla_c \nabla_b V_a - \nabla_b \nabla_c V_a = -\Gamma^d_{ab}\frac{\partial V_d}{\partial x^c} - V_d \frac{\partial \Gamma^d_{ab}}{\partial x^c} - \Gamma^e_{ac}\left(\frac{\partial V_e}{\partial x^b} - \Gamma^d_{be} V_d\right) + \Gamma^d_{ac}\frac{\partial V_d}{\partial x^b} + V_d \frac{\partial \Gamma^d_{ac}}{\partial x^b} + \Gamma^e_{ab}\left(\frac{\partial V_e}{\partial x^c} - \Gamma^d_{ce} V_d\right)$$

By expanding the parentheses in the last terms and factoring the terms with V_d :

$$\nabla_c \nabla_b V_a - \nabla_b \nabla_c V_a = -\Gamma^d_{ab}\frac{\partial V_d}{\partial x^c} - V_d \frac{\partial \Gamma^d_{ab}}{\partial x^c} - \Gamma^e_{ac}\frac{\partial V_e}{\partial x^b} + \Gamma^e_{ac}\Gamma^d_{be} V_d + \Gamma^d_{ac}\frac{\partial V_d}{\partial x^b} + V_d \frac{\partial \Gamma^d_{ac}}{\partial x^b} + \Gamma^e_{ab}\frac{\partial V_e}{\partial x^c} - \Gamma^e_{ab}\Gamma^d_{ce} V_d$$

$$= \Gamma^d_{ac}\frac{\partial V_d}{\partial x^b} - \Gamma^d_{ab}\frac{\partial V_d}{\partial x^c} + \Gamma^e_{ab}\frac{\partial V_e}{\partial x^c} - \Gamma^e_{ac}\frac{\partial V_e}{\partial x^b} + \left(\frac{\partial \Gamma^d_{ac}}{\partial x^b} - \frac{\partial \Gamma^d_{ab}}{\partial x^c} + \Gamma^e_{ac}\Gamma^d_{be} - \Gamma^e_{ab}\Gamma^d_{ce}\right) V_d$$

From [equation 2 5 1 1](#) in the previous chapter, we know:

$$\frac{\partial e_i}{\partial x^j} = \Gamma^k_{ij} e_k \qquad (3)$$

Therefore:

$$\frac{\partial V_e}{\partial x^c} = \Gamma^d_{ec} V_d \Rightarrow \Gamma^e_{ab}\frac{\partial V_e}{\partial x^c} = \Gamma^e_{ab}\Gamma^d_{ec} V_d \text{ en } \frac{\partial V_e}{\partial x^b} = \Gamma^d_{eb} V_d \Rightarrow \Gamma^e_{ac}\frac{\partial V_e}{\partial x^b} = \Gamma^e_{ac}\Gamma^d_{eb} V_d$$

$$\nabla_c \nabla_b V_a - \nabla_b \nabla_c V_a = \Gamma^d_{ac}\frac{\partial V_d}{\partial x^b} - \Gamma^d_{ab}\frac{\partial V_d}{\partial x^c} + \boxed{\cancel{\Gamma^e_{ab}\frac{\partial V_e}{\partial x^c}}} - \boxed{\cancel{\Gamma^e_{ac}\frac{\partial V_e}{\partial x^b}}} + \left(\frac{\partial \Gamma^d_{ac}}{\partial x^b} - \frac{\partial \Gamma^d_{ab}}{\partial x^c} \boxed{\cancel{+\Gamma^e_{ac}\Gamma^d_{be}}} - \boxed{\cancel{\Gamma^e_{ab}\Gamma^d_{ce}}}\right) V_d$$

$$\nabla_c \nabla_b V_a - \nabla_b \nabla_c V_a = \Gamma^d_{ac}\frac{\partial V_d}{\partial x^b} + V_d \frac{\partial \Gamma^d_{ac}}{\partial x^b} - \Gamma^d_{ab}\frac{\partial V_d}{\partial x^c} - V_d \frac{\partial \Gamma^d_{ab}}{\partial x^c}$$

After swapping *d* with *e* in the first and third terms on the right-hand side:

$$\nabla_c \nabla_b V_a - \nabla_b \nabla_c V_a = \Gamma^e_{ac}\frac{\partial V_e}{\partial x^b} + V_d \frac{\partial \Gamma^d_{ac}}{\partial x^b} - \Gamma^e_{ab}\frac{\partial V_e}{\partial x^c} - V_d \frac{\partial \Gamma^d_{ab}}{\partial x^c} =$$

$$\nabla_c \nabla_b V_a - \nabla_b \nabla_c V_a = \Gamma^e_{ac}\Gamma^d_{eb} V_d + V_d \frac{\partial \Gamma^d_{ac}}{\partial x^b} - \Gamma^e_{ab}\Gamma^d_{ec} V_d - V_d \frac{\partial \Gamma^d_{ab}}{\partial x^c} =$$

$$\nabla_c \nabla_b V_a - \nabla_b \nabla_c V_a = \left(\frac{\partial \Gamma^d_{ac}}{\partial x^b} - \frac{\partial \Gamma^d_{ab}}{\partial x^c} + \Gamma^e_{ac}\Gamma^d_{be} - \Gamma^e_{ab}\Gamma^d_{ce}\right) V_d$$

We define the expression inside the parentheses on the right-hand side as the **Riemann tensor**, which means that:

$$[\nabla_c, \nabla_b]V_a = \nabla_c\nabla_b V_a - \nabla_b\nabla_c V_a = R^d_{abc} V_d$$
$$R^d_{abc} = \frac{\partial \Gamma^d_{ac}}{\partial x^b} - \frac{\partial \Gamma^d_{ab}}{\partial x^c} + \Gamma^e_{ac}\Gamma^d_{be} - \Gamma^e_{ab}\Gamma^d_{ce}$$
$$R^d_{abc} = \Gamma^d_{ac,b} - \Gamma^d_{ab,c} + \Gamma^e_{ac}\Gamma^d_{be} - \Gamma^e_{ab}\Gamma^d_{ce}$$

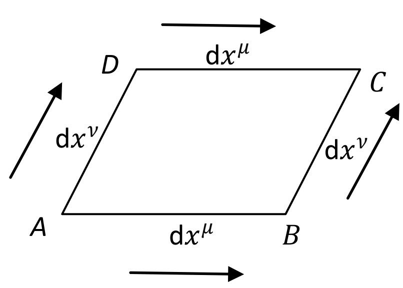
This is the **component form of the Riemann tensor**, which explicitly contains the derivatives of the Christoffel symbols and their products. This expression shows how curvature is an intrinsic geometric effect that cannot be removed by a change of coordinates.

Note:

Here, the commutator can be considered as the difference of two vectors. The magnitude of the resulting vector is the Riemann tensor.

2.10.1.3 Alternative Derivation of the Riemann Tensor via the Commutator

We consider an infinitesimal area over which a vector is moved (transported in parallel) via two different paths. When the manifold is flat, the difference between the two end vectors would be zero. However, in the case that the manifold is intrinsically curved, this would lead to a difference between the end vectors.
First, we move a vector $\overrightarrow{V}$ from point A via B to C. To determine the direction of the vector's movement, we take the derivative of the vector with respect to dx^μ and then look at the change in this result with respect to dx^ν .
Next we do the same from A via D to C, first with respect to dx^ν and then to dx^μ . Both results are subtracted from each other, which leads to the Riemann tensor.



$$\overrightarrow{V} = V^m \cdot \vec{e}_m$$

The vector $\vec{e}_m$ is the tangent vector, i.e., the derivative of the position vector or the derivative of the trajectory. If the trajectory is a straight line, then the derivative of $\vec{e}_m$ is a constant; and consequently, the derivative of $\vec{e}_m$, and thus the Christoffel symbol, is zero.

First, from A to B to determine the direction, we take the derivative (see also equation 3):

$$\frac{\partial \overrightarrow{V}}{\partial x^{\mu}} = \frac{\partial V^m}{\partial x^{\mu}} \cdot \vec{e}_m + V^m \frac{\partial \vec{e}_m}{\partial x^{\mu}} = \frac{\partial V^m}{\partial x^{\mu}} \cdot \vec{e}_m + V^m \Gamma^k_{m\mu} \vec{e}_k$$

Change the two dummy indices, *k* and *m*. Then the formula can be adapted from *k* to *m* and *m* to γ .

$$\frac{\partial \vec{V}}{\partial x^{\mu}} = \frac{\partial V^m}{\partial x^{\mu}} \vec{e}_m + V^{\gamma} \Gamma^m_{\gamma\mu} \vec{e}_m = \left(\frac{\partial V^m}{\partial x^{\mu}} + V^{\gamma} \Gamma^m_{\gamma\mu} \right) \vec{e}_m$$

This is the covariant derivative of the contravariant vector $\vec{V}$. And from the definition of the Christoffel symbol in the previous chapters, we know that $\frac{\partial \vec{e}_m}{\partial x^{\mu}} = \Gamma^k_{m\mu} \vec{e}_k$ (see also equation 3).
Next, change the direction from B to C with respect to dx^{ν} :

$$\frac{\partial^2 \vec{V}}{\partial x^{\nu}\, \partial x^{\mu}} = \frac{\partial^2 V^m}{\partial x^{\nu}\, \partial x^{\mu}} \vec{e}_m + \frac{\partial V^m}{\partial x^{\mu}} \frac{\vec{e}_m}{\partial x^{\nu}} + \frac{\partial V^{\gamma}}{\partial x^{\nu}} \Gamma^m_{\gamma\mu} \vec{e}_m + V^{\gamma} \frac{\partial \Gamma^m_{\gamma\mu}}{\partial x^{\nu}} \vec{e}_m + V^{\gamma} \Gamma^m_{\gamma\mu} \frac{\partial \vec{e}_m}{\partial x^{\nu}}$$

$$\frac{\partial^2 \vec{V}}{\partial x^{\nu}\, \partial x^{\mu}} = \frac{\partial^2 V^m}{\partial x^{\nu}\, \partial x^{\mu}} \vec{e}_m + \frac{\partial V^m}{\partial x^{\mu}} \Gamma^k_{m\nu} \vec{e}_k + \frac{\partial V^{\gamma}}{\partial x^{\nu}} \Gamma^m_{\gamma\mu} \vec{e}_m + V^{\gamma} \frac{\partial \Gamma^m_{\gamma\mu}}{\partial x^{\nu}} \vec{e}_m + V^{\gamma} \Gamma^m_{\gamma\mu} \Gamma^k_{m\nu} \vec{e}_k$$

In the right-hand side of the equation, replace the indices *k* with *m* and *m* with γ in the second term, and swap *k* and *m* in the fifth term:

$$\frac{\partial^2 \vec{V}}{\partial x^{\nu}\, \partial x^{\mu}} = \frac{\partial^2 V^m}{\partial x^{\nu}\, \partial x^{\mu}} \vec{e}_m + \frac{\partial V^{\gamma}}{\partial x^{\mu}} \Gamma^m_{\gamma\nu} \vec{e}_m + \frac{\partial V^{\gamma}}{\partial x^{\nu}} \Gamma^m_{\gamma\mu} \vec{e}_m + V^{\gamma} \frac{\partial \Gamma^m_{\gamma\mu}}{\partial x^{\nu}} \vec{e}_m + V^{\gamma} \Gamma^k_{\gamma\mu} \Gamma^m_{k\nu} \vec{e}_m$$

$$\frac{\partial^2 \vec{V}}{\partial x^{\nu}\, \partial x^{\mu}} = \frac{\partial \Gamma^m_{\gamma\mu}}{\partial x^{\nu}} V^{\gamma} \vec{e}_m + \Gamma^k_{\gamma\mu} \Gamma^m_{k\nu} V^{\gamma} \vec{e}_m + \frac{\partial^2 V^m}{\partial x^{\nu}\, \partial x^{\mu}} \vec{e}_m + \frac{\partial V^{\gamma}}{\partial x^{\mu}} \Gamma^m_{\gamma\nu} \vec{e}_m + \frac{\partial V^{\gamma}}{\partial x^{\nu}} \Gamma^m_{\gamma\mu} \vec{e}_m$$

Now for the other direction, swap μ and ν :

$$\frac{\partial^2 \vec{V}}{\partial x^{\mu}\, \partial x^{\nu}} = \frac{\partial \Gamma^m_{\gamma\nu}}{\partial x^{\mu}} V^{\gamma} \vec{e}_m + \Gamma^k_{\gamma\nu} \Gamma^m_{k\mu} V^{\gamma} \vec{e}_m + \frac{\partial^2 V^m}{\partial x^{\mu}\, \partial x^{\nu}} \vec{e}_m + \frac{\partial V^{\gamma}}{\partial x^{\nu}} \Gamma^m_{\gamma\mu} \vec{e}_m + \frac{\partial V^{\gamma}}{\partial x^{\mu}} \Gamma^m_{\gamma\nu} \vec{e}_m$$

Now subtract the last two equations from each other:

$$\frac{\partial^2 \vec{V}}{\partial x^{\mu}\, \partial x^{\nu}} - \frac{\partial^2 \vec{V}}{\partial x^{\nu}\, \partial x^{\mu}} =$$

$$= \frac{\partial \Gamma^m_{\gamma\nu}}{\partial x^{\mu}} V^{\gamma} \vec{e}_m - \frac{\partial \Gamma^m_{\gamma\mu}}{\partial x^{\nu}} V^{\gamma} \vec{e}_m + \Gamma^k_{\gamma\nu} \Gamma^m_{k\mu} V^{\gamma} \vec{e}_m - \Gamma^k_{\gamma\mu} \Gamma^m_{k\nu} V^{\gamma} \vec{e}_m + \frac{\partial^2 V^m}{\partial x^{\mu}\, \partial x^{\nu}} \vec{e}_m - \frac{\partial^2 V^m}{\partial x^{\nu}\, \partial x^{\mu}} \vec{e}_m + \frac{\partial V^{\gamma}}{\partial x^{\nu}} \Gamma^m_{\gamma\mu} \vec{e}_m$$
$$- \frac{\partial V^{\gamma}}{\partial x^{\nu}} \Gamma^m_{\gamma\mu} \vec{e}_m + \frac{\partial V^{\gamma}}{\partial x^{\mu}} \Gamma^m_{\gamma\nu} \vec{e}_m - \frac{\partial V^{\gamma}}{\partial x^{\mu}} \Gamma^m_{\gamma\nu} \vec{e}_m$$

$$= \frac{\partial \Gamma^m_{\gamma\nu}}{\partial x^{\mu}} V^{\gamma} \vec{e}_m - \frac{\partial \Gamma^m_{\gamma\mu}}{\partial x^{\nu}} V^{\gamma} \vec{e}_m + \Gamma^k_{\gamma\nu} \Gamma^m_{k\mu} V^{\gamma} \vec{e}_m - \Gamma^k_{\gamma\mu} \Gamma^m_{k\nu} V^{\gamma} \vec{e}_m$$

$$\Longrightarrow \frac{\partial^2 \vec{V}}{\partial x^{\mu}\, \partial x^{\nu}} - \frac{\partial^2 \vec{V}}{\partial x^{\nu}\, \partial x^{\mu}} = \left(\boldsymbol{\frac{\partial \Gamma^m_{\gamma\nu}}{\partial x^{\mu}} - \frac{\partial \Gamma^m_{\gamma\mu}}{\partial x^{\nu}} + \Gamma^k_{\gamma\nu} \Gamma^m_{k\mu} - \Gamma^k_{\gamma\mu} \Gamma^m_{k\nu}} \right) V^{\gamma} \vec{e}_m$$

$$\frac{\partial^2 \vec{V}}{\partial x^{\mu}\, \partial x^{\nu}} - \frac{\partial^2 \vec{V}}{\partial x^{\nu}\, \partial x^{\mu}} = \boldsymbol{R^m_{\gamma\mu\nu}} V^{\gamma} \vec{e}_m$$

2.10.1.4 Definition of the Riemann tensor:

The expression within the brackets is defined as the Riemann tensor $\boldsymbol{R^m_{\gamma\mu\nu}}$ **:**

$$R^{m}_{\gamma\mu\nu} = \frac{\partial\Gamma^{m}_{\gamma\nu}}{\partial x^{\mu}} - \frac{\partial\Gamma^{m}_{\gamma\mu}}{\partial x^{\nu}} + \Gamma^{k}_{\gamma\nu}\Gamma^{m}_{k\mu} - \Gamma^{k}_{\gamma\mu}\Gamma^{m}_{k\nu}$$

Where the Riemann tensor describes the degree of curvature of space-time by the difference in parallel transport of a tensor around a closed loop.

2.10.1.5 Conclusion:

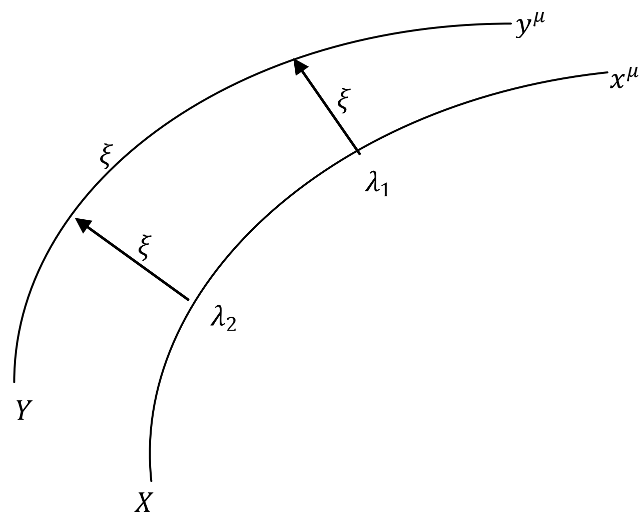
This alternative derivation of the Riemann tensor via the commutator provides a way to understand how the curvature of space-time is determined by the difference in parallel transport of tensors. The Riemann tensor is therefore a crucial tool in general relativity for describing the geometry and gravitational effects in space-time.

2.10.2 Derivation of the Riemann Tensor via Geodesic Deviation

In the previous chapter, we showed a method for deriving the Riemann tensor from the commutator of covariant derivatives, which physically corresponds to the difference between transporting a vector in parallel first along one path and then along another. Another interpretation arises from the relative acceleration of nearby particles in free fall.

Imagine a cloud of particles in free fall. Let us assume that an observer is traveling with one of these particles. He looks at a nearby particle and measures its position in local inertial coordinates. In Special Relativity, this particle will move in a straight line at constant speed, without acceleration. But what happens in a gravitational field?

As we remember from the previous chapter, a geodesic line generalizes the concept of a "straight line" to curved space-time.
Here we will show how the evolution of the distance measured between two neighboring geodesic lines, also called geodesic deviation, can indeed be related to a non-zero curvature of space-time, or in Newtonian terms, to the presence of *tidal forces*. So let's consider two particles following two very close geodesic lines.
Their respective paths can be described by the functions $x^{\mu}(\tau)$ (for the reference particle) and $y^{\mu}(\tau) \equiv x^{\mu}(\tau) + \xi^{\mu}(\tau)$ (for the second particle), where τ (tau) is the proper time along the worldline of the reference particle, and where ξ^{μ} refers to the deviation four-vector connecting one particle to the other at any given moment τ .



The relative acceleration A^α of the two objects is roughly defined as the second derivative of the separation vector ξ^α as the objects move along their respective geodesics.

Our goal in this chapter is to show that this relative acceleration is related to the Riemann tensor via the following equation:

$$\left(\frac{d^2\xi}{d\tau^2}\right)^\alpha = -R^\alpha_{\mu\sigma\nu}\, u^\nu u^\mu \xi^\sigma$$

In the case where space-time is flat, the Riemann tensor is zero, resulting in zero relative acceleration.

Since each particle follows a geodesic line, the equation of their respective coordinates is as follows (see equation_2_6_1):

$$0 = \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\left(x^\alpha(\tau)\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$
$$0 = \frac{d^2y^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\left(y^\alpha(\tau)\right)\frac{dy^\mu}{d\tau}\frac{dy^\nu}{d\tau}$$

In each of these equations, the Christoffel symbol is equal at each respective position of the particles *x* and *y*. Since the separation between the particles is infinitesimal, we evaluate the Christoffel symbol at the position $y^\alpha(\tau)$ using a Taylor series expansion:

$$f(x) = f(a) + \frac{f^{'}(a)}{1!}(x-a) + \frac{f^{''}(a)}{2!}(x-a)^2 \dots \frac{f^n(a)}{n!}(x-a)^n$$

Approximating the first derivative because ξ is infinitesimal.

$$\Gamma^\alpha_{\mu\nu}\left(y^\alpha(\tau)\right) \approx \Gamma^\alpha_{\mu\nu}\left(x^\alpha(\tau)\right) + \xi^\sigma\left[\partial_\sigma\Gamma^\alpha_{\mu\nu}\left(x^\alpha(\tau)\right)\right]$$

This can also be approximated as follows for an infinitesimal Δx:

$$\frac{d\Gamma^{\alpha}_{\mu\nu}(x)}{dx} = \frac{\Gamma^{\alpha}_{\mu\nu}(x+\Delta x) - \Gamma^{\alpha}_{\mu\nu}(x)}{\Delta x}$$

$$\Gamma^{\alpha}_{\mu\nu}(x+\Delta x) = \Gamma^{\alpha}_{\mu\nu}(x) + \Delta x \frac{d\Gamma^{\alpha}_{\mu\nu}(x)}{dx}$$

$$\Delta x = \xi$$

$$\Gamma^{\alpha}_{\mu\nu}(x+\xi) = \Gamma^{\alpha}_{\mu\nu}(x) + \xi \frac{d\Gamma^{\alpha}_{\mu\nu}(x)}{dx}$$

Assuming that $y^{\alpha}(\tau) = x^{\alpha}(\tau) + \xi^{\alpha}(\tau)$ and substituting this last expression into the geodesic equation of particles *y*, we obtain:

$$0 = \frac{d^2 y^{\alpha}}{d\tau^2} + \Gamma^{\alpha}_{\mu\nu}\left(y^{\alpha}(\tau)\right)\frac{dy^{\mu}}{d\tau}\frac{dy^{\nu}}{d\tau}$$

$$0 = \frac{d^2(x^{\alpha}+\xi^{\alpha})}{d\tau^2} + \left[\Gamma^{\alpha}_{\mu\nu} + \xi^{\sigma}\left(\partial_{\sigma}\Gamma^{\alpha}_{\mu\nu}\right)\right]\frac{d(x^{\mu}+\xi^{\mu})}{d\tau}\frac{d(x^{\nu}+\xi^{\nu})}{d\tau}$$

$$0 = \frac{d^2 x^{\alpha}}{d\tau^2} + \frac{d^2 \xi^{\alpha}}{d\tau^2} + \left[\Gamma^{\alpha}_{\mu\nu} + \xi^{\sigma}\left(\partial_{\sigma}\Gamma^{\alpha}_{\mu\nu}\right)\right]\left(\frac{dx^{\mu}}{d\tau} + \frac{d\xi^{\mu}}{d\tau}\right)\left(\frac{dx^{\nu}}{d\tau} + \frac{d\xi^{\nu}}{d\tau}\right)$$

Here, the Christoffel symbol and its first-order derivatives are now evaluated at $x^{\alpha}(\tau)$.
By working out all the terms in the brackets and neglecting the second-order terms with respect to ξ, we obtain:

$$0 = \frac{d^2 x^{\alpha}}{d\tau^2} + \frac{d^2 \xi^{\alpha}}{d\tau^2} + \Gamma^{\alpha}_{\mu\nu}\left(\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} + \frac{dx^{\mu}}{d\tau}\frac{d\xi^{\nu}}{d\tau} + \frac{d\xi^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} + \cancel{\frac{d\xi^{\mu}}{d\tau}\frac{d\xi^{\nu}}{d\tau}}\right) +$$

$$+\xi^{\sigma}\left(\partial_{\sigma}\Gamma^{\alpha}_{\mu\nu}\right)\left(\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} + \cancel{\frac{dx^{\mu}}{d\tau}\frac{d\xi^{\nu}}{d\tau}} + \cancel{\frac{d\xi^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}} + \cancel{\frac{d\xi^{\mu}}{d\tau}\frac{d\xi^{\nu}}{d\tau}}\right)$$

Since we know that the Christoffel symbol is symmetric with respect to the lower indices, these can be interchanged:

$$0 = \frac{d^2 x^{\alpha}}{d\tau^2} + \frac{d^2 \xi^{\alpha}}{d\tau^2} + \Gamma^{\alpha}_{\mu\nu}\left(\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} + 2\frac{dx^{\mu}}{d\tau}\frac{d\xi^{\nu}}{d\tau}\right) + \xi^{\sigma}\left(\partial_{\sigma}\Gamma^{\alpha}_{\mu\nu}\right)\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}$$

Using the geodesic equation of particle *x*, as given (see equation_2_6_1):

$$\frac{d^2 x^{\alpha}}{d\tau^2} = -\Gamma^{\alpha}_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}$$

Then the first and third terms drop out. Then we obtain:

$$0 = \frac{d^2 \xi^{\alpha}}{d\tau^2} + 2\Gamma^{\alpha}_{\mu\nu}u^{\mu}\frac{d\xi^{\nu}}{d\tau} + \xi^{\sigma}\left(\partial_{\sigma}\Gamma^{\alpha}_{\mu\nu}\right)u^{\mu}u^{\nu}$$

$$\frac{d^2 \xi^{\alpha}}{d\tau^2} = -2\Gamma^{\alpha}_{\mu\nu}u^{\mu}\frac{d\xi^{\nu}}{d\tau} - \xi^{\sigma}\left(\partial_{\sigma}\Gamma^{\alpha}_{\mu\nu}\right)u^{\mu}u^{\nu}$$

Here, $u^{\mu} = \frac{dx^{\mu}}{d\tau}$ is the four-velocity vector of the reference particle.

We then have an expression for $\frac{d\xi^\alpha}{d\tau}$, but this is not the total derivative of the four-vector ξ, since the derivative can also receive a contribution from the change in the basis vectors as the object moves along its geodesic line. To obtain the total derivative, we use:

$$\frac{d\xi}{d\tau} = \frac{d}{d\tau}(\xi^\alpha \boldsymbol{e}_\alpha) = \frac{d\xi^\alpha}{d\tau}\boldsymbol{e}_\alpha + \xi^\alpha \frac{d\boldsymbol{e}_\alpha}{d\tau} = \frac{d\xi^\alpha}{d\tau}\boldsymbol{e}_\alpha + \xi^\alpha \frac{dx^\mu}{d\tau}\frac{d\boldsymbol{e}_\alpha}{dx^\mu}$$

By replacing the dummy index α with σ in the second term and using the definition of the Christoffel symbol, we obtain:

$$\xi^\sigma \frac{dx^\mu}{d\tau}\frac{d\boldsymbol{e}_\sigma}{dx^\mu} = \xi^\sigma \frac{dx^\mu}{d\tau}\Gamma^\alpha_{\mu\sigma}\boldsymbol{e}_\alpha = \xi^\sigma u^\mu \Gamma^\alpha_{\mu\sigma}\boldsymbol{e}_\alpha$$

$$\Rightarrow \frac{d\xi}{d\tau} = \frac{d\xi^\alpha}{d\tau}\boldsymbol{e}_\alpha + \xi^\sigma u^\mu \Gamma^\alpha_{\mu\sigma}\boldsymbol{e}_\alpha = \left(\frac{d\xi^\alpha}{d\tau} + \Gamma^\alpha_{\mu\sigma}\xi^\sigma u^\mu\right)\boldsymbol{e}_\alpha$$

So that:

$$\left(\frac{d\xi}{d\tau}\right)^\alpha = \frac{d\xi^\alpha}{d\tau} + \Gamma^\alpha_{\mu\sigma}\xi^\sigma u^\mu$$

Since we are still dealing with the condition that ξ is a four-vector, its derivative with respect to proper time is also a four-vector, so we can find the second absolute derivative by using the same development as for the first-order derivative.

$$\left(\frac{d}{d\tau}\left[\frac{d\xi}{d\tau}\right]\right)^\alpha = \frac{d}{d\tau}\left(\left[\frac{d\xi}{d\tau}\right]^\alpha\right) + \Gamma^\alpha_{\mu\sigma}\left(\frac{d\xi}{d\tau}\right)^\sigma u^\mu$$

$$\left(\frac{d^2\xi}{d\tau^2}\right)^\alpha = \left(\frac{d}{d\tau}\left[\frac{d\xi}{d\tau}\right]\right)^\alpha = \frac{d}{d\tau}\left(\frac{d\xi^\alpha}{d\tau} + \Gamma^\alpha_{\mu\sigma}u^\mu\xi^\sigma\right) + \Gamma^\alpha_{\mu\sigma}u^\mu\left(\frac{d\xi^\sigma}{d\tau} + \Gamma^\sigma_{\beta\gamma}u^\beta\xi^\gamma\right)$$

$$= \frac{d^2\xi^\alpha}{d\tau^2} + \frac{d\Gamma^\alpha_{\mu\sigma}}{d\tau}u^\mu\xi^\sigma + \Gamma^\alpha_{\mu\sigma}\frac{du^\mu}{d\tau}\xi^\sigma + \Gamma^\alpha_{\mu\sigma}u^\mu\frac{d\xi^\sigma}{d\tau} + \Gamma^\alpha_{\mu\sigma}u^\mu\frac{d\xi^\sigma}{d\tau} + \Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\beta\gamma}u^\mu u^\beta\xi^\gamma$$

By using the Christoffel symbols and Taylor series above and exchange ν by σ in the **first term**, we obtain:

$$\frac{d^2\xi^\alpha}{d\tau^2} = -2\Gamma^\alpha_{\mu\nu}\mathrm{u}^\mu\frac{d\xi^\nu}{d\tau} - \left(\frac{d\Gamma^\alpha_{\mu\nu}}{dx^\sigma}\right)u^\mu u^\nu\xi^\sigma$$

Exchange in the first term, at the right hand side, ν with σ:

$$\frac{d^2\xi^\alpha}{d\tau^2} = -2\Gamma^\alpha_{\mu\sigma}\mathrm{u}^\mu\frac{d\xi^\sigma}{d\tau} - \left(\frac{d\Gamma^\alpha_{\mu\nu}}{dx^\sigma}\right)u^\mu u^\nu\xi^\sigma$$

We can rewrite the **second term**, since the Christoffel symbols depend on τ by depending on the position of the reference particle:

$$\Rightarrow \frac{d\Gamma^\alpha_{\mu\sigma}}{d\tau}u^\mu\xi^\sigma = \frac{d\Gamma^\alpha_{\mu\sigma}}{dx^\nu}\frac{dx^\nu}{d\tau}u^\mu\xi^\sigma = \frac{d\Gamma^\alpha_{\mu\sigma}}{dx^\nu}u^\nu u^\mu\xi^\sigma$$

By using the geodesic equation, we can rewrite the **third term**, i.e., work out $\frac{du^\mu}{d\tau}$:

$$u^\mu = \frac{dx^\mu}{d\tau}$$

$$\frac{du^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2}$$

$$Geodesic\ equation: \quad \frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\nu\gamma}\frac{dx^\nu}{d\tau}\frac{dx^\gamma}{d\tau} = -\Gamma^\mu_{\nu\gamma}\,u^\nu u^\gamma = \frac{du^\mu}{d\tau}$$

$$\Longrightarrow \Gamma^\alpha_{\mu\sigma}\frac{du^\mu}{d\tau}\xi^\sigma = -\Gamma^\alpha_{\mu\sigma}\Gamma^\mu_{\nu\gamma}\,u^\nu u^\gamma \xi^\sigma$$

Swap, in the right hand term, μ with γ:

$$\Gamma^\alpha_{\mu\sigma}\frac{du^\mu}{d\tau}\xi^\sigma = -\Gamma^\alpha_{\gamma\sigma}\Gamma^\gamma_{\nu\mu}\,u^\nu u^\mu \xi^\sigma$$

Also, to obtain an expression for $u^\nu u^\mu \xi^\sigma$, with only μ, ν and σ, we can rewrite the **last term** by renaming the dummy indices σ and β :

$$\Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\beta\gamma}\,u^\mu u^\beta \xi^\gamma =$$

$$(\sigma \leftrightarrow \gamma) = \Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\beta\sigma}\,u^\mu u^\beta \xi^\sigma$$

$$(\beta \leftrightarrow \nu) = \Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\nu\sigma}\,u^\mu u^\nu \xi^\sigma$$

$$(\mu \leftrightarrow \nu) = \Gamma^\alpha_{\nu\gamma}\Gamma^\gamma_{\mu\sigma}\,u^\nu u^\mu \xi^\sigma$$

So finally, by replacing all terms, we can write:

$$\left(\frac{d^2\xi}{d\tau^2}\right)^\alpha = \frac{d^2\xi^\alpha}{d\tau^2} + \frac{d\Gamma^\alpha_{\mu\sigma}}{d\tau}u^\mu \xi^\sigma + \Gamma^\alpha_{\mu\sigma}\frac{du^\mu}{d\tau}\xi^\sigma + \left(\Gamma^\alpha_{\mu\sigma}u^\mu\frac{d\xi^\sigma}{d\tau} + \Gamma^\alpha_{\mu\sigma}u^\mu\frac{d\xi^\sigma}{d\tau}\right) + \Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\beta\gamma}\,u^\mu u^\beta \xi^\gamma$$

$$= \cancel{-2\Gamma^\alpha_{\mu\sigma}u^\mu\frac{d\xi^\sigma}{d\tau}} - \left(\frac{d\Gamma^\alpha_{\mu\nu}}{dx^\sigma}\right)u^\mu u^\nu \xi^\sigma + \frac{d\Gamma^\alpha_{\mu\sigma}}{dx^\nu}u^\nu u^\mu \xi^\sigma - \Gamma^\alpha_{\gamma\sigma}\Gamma^\gamma_{\nu\mu}\,u^\nu u^\mu \xi^\sigma + \cancel{2\Gamma^\alpha_{\mu\sigma}u^\mu\frac{d\xi^\sigma}{d\tau}} + \Gamma^\alpha_{\nu\gamma}\Gamma^\gamma_{\mu\sigma}\,u^\nu u^\mu \xi^\sigma$$

By deleting the first and fifth terms and extracting the common factor $u^\nu u^\mu \xi^\sigma$, we obtain:

$$\left(\frac{d^2\xi}{d\tau^2}\right)^\alpha = -\left(\frac{d\Gamma^\alpha_{\mu\nu}}{dx^\sigma} - \frac{d\Gamma^\alpha_{\mu\sigma}}{dx^\nu} + \Gamma^\alpha_{\gamma\sigma}\Gamma^\gamma_{\nu\mu} - \Gamma^\alpha_{\nu\gamma}\Gamma^\gamma_{\mu\sigma}\right)u^\nu u^\mu \xi^\sigma$$

Since this is still a tensor equation, the quantity in parentheses is a tensor, and we can define the ***Riemann tensor*** as:

$$\boxed{\boldsymbol{R^\alpha_{\mu\sigma\nu} = \left(\frac{d\Gamma^\alpha_{\mu\nu}}{dx^\sigma} - \frac{d\Gamma^\alpha_{\mu\sigma}}{dx^\nu} + \Gamma^\alpha_{\sigma\gamma}\Gamma^\gamma_{\mu\nu} - \Gamma^\alpha_{\nu\gamma}\Gamma^\gamma_{\mu\sigma}\right)}}$$

Then we can rewrite the above equation in a shorter expression, known as ***the geodesic deviation equation***:

$$\boldsymbol{\left(\frac{d^2\xi}{d\tau^2}\right)^\alpha = -R^\alpha_{\mu\sigma\nu}u^\nu u^\mu \xi^\sigma}$$

Since the only quantity in this equation that is intrinsically dependent on the metric is the Riemann tensor, we see that if it is identical to zero, then space-time is flat. But if just one component of this tensor is nonzero, then space-time is curved.

2.10.3 Key insights

- The **Riemann tensor** $R^\rho_{\sigma\mu\nu}$ is the fundamental tensor that describes **the curvature of space-time**.

- It can be derived via the commutator of covariant derivatives, or via the geodesic deviation equation.
- Its component form is:

$$R^{\alpha}_{\mu\sigma\nu} = \frac{d\Gamma^{\alpha}_{\mu\nu}}{dx^{\sigma}} - \frac{d\Gamma^{\alpha}_{\mu\sigma}}{dx^{\nu}} + \Gamma^{\alpha}_{\gamma\sigma}\Gamma^{\gamma}_{\nu\mu} - \Gamma^{\alpha}_{\nu\gamma}\Gamma^{\gamma}_{\mu\sigma}$$

- The following property applies to a geodesic line:

$$0 = \frac{d^2x^{\beta}}{d\tau^2} + \Gamma^{\beta}_{\mu\nu}\frac{\partial x^{\mu}}{\partial\tau}\frac{\partial x^{\nu}}{\partial\tau} \quad \textit{Geodesic equation}$$

- Or:

$$\frac{d^2x^{\beta}}{d\tau^2} = -\boldsymbol{\Gamma^{\beta}_{\mu\nu}}u^{\nu}u^{\mu}$$

- While for the deviation from one geodesic to an infinitesimally nearby geodesic line, the following applies:

$$\left(\frac{d^2\xi}{d\tau^2}\right)^{\alpha} = -\boldsymbol{R^{\alpha}_{\mu\sigma\nu}}u^{\nu}u^{\mu}\xi^{\sigma} \quad \text{Geodesic deviation equation}$$

- A non-zero Riemann tensor implies curved space-time and thus the presence of gravity.

- Measure the **non-commutativity effect** of twice covariant differentiation of a vector:

$$\left(\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu}\right)V^{\rho} = R^{\rho}_{\sigma\mu\nu}V^{\sigma}$$

The tensor can be expressed entirely in terms of **Christoffel symbols and their derivatives**.

- In flat space, $R^{\rho}_{\sigma\mu\nu} = 0$; in curved space, it is generally non-zero.
- Curvature can **be measured locally** via the behavior of geodesics: if two free particles that start close to each other begin to diverge, this indicates curvature.

2.10.4 Intuitive explanation

Imagine two rockets starting to fly side by side in space, without engines (free falling), each in a slightly different position.
In flat space, they remain **parallel**, but in curved space (e.g., around a planet), they will **bend toward or away from each other**.

The **Riemann tensor** measures precisely that effect:

- How does the "direction" of a vector change when you transport it in a closed loop?
- If the result differs from the original vector, the space is **curved**.

You can compare it to an arrow that you take on a tour around a sphere: after returning, it no longer points in the same **direction—curvature manifests itself as a change in direction**.

Table overview:

Quantity	**Meaning**
$R^{\rho}_{\sigma\mu\nu}$	Measure curvature by comparing transport
Building blocks	Christoffel symbols + their derivatives
Physical meaning	Deviation between nearby geodesics
Zero in flat space	$R^{\rho}_{\sigma\mu\nu} = 0$
Dimension	4th-rank tensor (4 indices)

2.11 Symmetries and independent components

In the previous chapters, we derived the rather complex expression for the Riemann curvature tensor—a combination of derivatives and products of Christoffel symbols, with a total of 256 (=4^4) components in a four-dimensional space-time. In this chapter, we show that the Riemann tensor actually has only 20 independent components, and that these are completely determined by the symmetries of the tensor and the second-order derivatives of the metric.

We investigate these symmetries in a Local Inertial Frame (LIF), in which all Christoffel symbols are zero at the origin. However, these symmetries are not limited to this specific system: because tensor equations are coordinate-independent, they apply in every reference frame.

2.11.1 Definition and Reformulation

The Riemann tensor is generally defined as:

$$R^{\alpha}_{\beta\mu\nu} \equiv \frac{d\Gamma^{\alpha}_{\beta\nu}}{dx^{\mu}} - \frac{d\Gamma^{\alpha}_{\beta\mu}}{dx^{\nu}} + \Gamma^{\alpha}_{\mu\gamma}\,\Gamma^{\gamma}_{\beta\nu} - \Gamma^{\alpha}_{\nu\gamma}\,\Gamma^{\gamma}_{\beta\mu}$$

Knowing that all Christoffel symbols, $\Gamma = 0$, are zero at the origin of the Local Inertial Frame, this reduces to:

$$R^{\alpha}_{\beta\mu\nu} \equiv \frac{d\Gamma^{\alpha}_{\beta\nu}}{dx^{\mu}} - \frac{d\Gamma^{\alpha}_{\beta\mu}}{dx^{\nu}}$$

By applying the contraction mechanism, we can rewrite the Riemann tensor with all indices lowered:

$$R_{\alpha\beta\mu\nu} \equiv g_{\alpha\sigma}\,R^{\sigma}_{\beta\mu\nu} \equiv g_{\alpha\sigma}\left[\frac{d\Gamma^{\sigma}_{\beta\nu}}{dx^{\mu}} - \frac{d\Gamma^{\sigma}_{\beta\mu}}{dx^{\nu}}\right]$$

The Christoffel symbols can be expressed in terms of the metric:

$$\Gamma^{\sigma}_{\beta\nu} = \frac{1}{2}g^{\sigma\gamma}\left(\frac{\partial g_{\nu\gamma}}{\partial \mathrm{x}^{\beta}} + \frac{\partial g_{\gamma\beta}}{\partial \mathrm{x}^{\nu}} - \frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^{\gamma}}\right)$$

So we can write:

$$g_{\alpha\sigma}\frac{d\Gamma^{\sigma}_{\beta\nu}}{dx^{\mu}} = \frac{1}{2}g_{\alpha\sigma}\,g^{\sigma\gamma}\left(\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\nu\gamma}}{\partial \mathrm{x}^{\beta}} + \frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\gamma\beta}}{\partial \mathrm{x}^{\nu}} - \frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^{\gamma}}\right) + \frac{1}{2}g_{\alpha\sigma}\frac{\partial g^{\sigma\gamma}}{\partial x^{\mu}}\left(\frac{\partial g_{\nu\gamma}}{\partial \mathrm{x}^{\beta}} + \frac{\partial g_{\gamma\beta}}{\partial \mathrm{x}^{\nu}} - \frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^{\gamma}}\right) \quad (1)$$

The **second term** is zero because the Christoffel symbols are zero at the origin of the local inertial frame, as mentioned above:

$$\frac{1}{2}g_{\alpha\sigma}\frac{\partial g^{\sigma\gamma}}{\partial x^{\mu}}\left(\frac{\partial g_{\nu\gamma}}{\partial \mathrm{x}^{\beta}} + \frac{\partial g_{\gamma\beta}}{\partial \mathrm{x}^{\nu}} - \frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^{\gamma}}\right) =$$

$$= g_{\alpha\sigma}\frac{\partial g^{\sigma\gamma}}{\partial x^{\mu}}g_{\sigma\gamma}\,\frac{1}{2}g^{\sigma\gamma}\left(\frac{\partial g_{\nu\gamma}}{\partial \mathrm{x}^{\beta}} + \frac{\partial g_{\gamma\beta}}{\partial \mathrm{x}^{\nu}} - \frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^{\gamma}}\right) =$$

$$= g_{\alpha\sigma}\frac{\partial g^{\sigma\gamma}}{\partial x^{\mu}}g_{\sigma\gamma}\,\Gamma^{\sigma}_{\beta\nu} = 0$$

With this result and from equation (1), it follows that:

$$g_{\alpha\sigma}\frac{d\Gamma^{\sigma}_{\beta\nu}}{dx^{\mu}} = \frac{1}{2}\delta^{\gamma}_{\alpha}\left(\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\nu\gamma}}{\partial \mathrm{x}^{\beta}} + \frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\gamma\beta}}{\partial \mathrm{x}^{\nu}} - \frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^{\gamma}}\right)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\nu\alpha}}{\partial \mathrm{x}^{\beta}} + \frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\alpha\beta}}{\partial \mathrm{x}^{\nu}} - \frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^{\alpha}}\right)$$

Swapping indices μ and ν leads to the second term of the expression of the Riemann tensor:

$$g_{\alpha\sigma}\frac{d\Gamma^{\sigma}_{\beta\mu}}{dx^{\nu}} = \frac{1}{2}\left(\frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\mu\alpha}}{\partial \mathrm{x}^{\beta}} + \frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\alpha\beta}}{\partial \mathrm{x}^{\mu}} - \frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\beta\mu}}{\partial \mathrm{x}^{\alpha}}\right)$$

The middle terms disappear after subtracting the last two expressions, resulting in:

$$R_{\alpha\beta\mu\nu} = g_{\alpha\sigma}\left[\frac{d\Gamma^{\sigma}_{\beta\nu}}{dx^{\mu}} - \frac{d\Gamma^{\sigma}_{\beta\mu}}{dx^{\nu}}\right]$$

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left[\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\nu\alpha}}{\partial \mathrm{x}^{\beta}} + \frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\beta\mu}}{\partial \mathrm{x}^{\alpha}} - \frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\mu\alpha}}{\partial \mathrm{x}^{\beta}} - \frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^{\alpha}}\right] \quad (2)$$

Multiplied by -1:

$$R_{\alpha\beta\mu\nu} = -\frac{1}{2}\left[\frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\mu\alpha}}{\partial \mathrm{x}^{\beta}} + \frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^{\alpha}} - \frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\nu\alpha}}{\partial \mathrm{x}^{\beta}} - \frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\beta\mu}}{\partial \mathrm{x}^{\alpha}}\right] \quad (3)$$

Swapping μ and ν in (2):

$$R_{\alpha\beta\nu\mu} = \frac{1}{2}\left[\frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\mu\alpha}}{\partial \mathrm{x}^{\beta}} + \frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^{\alpha}} - \frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\nu\alpha}}{\partial \mathrm{x}^{\beta}} - \frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\beta\mu}}{\partial \mathrm{x}^{\alpha}}\right] \quad (4)$$

So, from (3) and (4) we get:

$$\boxed{R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu}}$$

Note that this equation is only valid at the origin of the Local Inertial Frame. But since these are tensor equations and, as we know, if these tensor equations are valid in one reference frame, they are valid in every reference frame.

Now we will demonstrate in a similar way that the Riemann tensor is anti-symmetric by swapping the first two indices:

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left[\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\nu\alpha}}{\partial \mathrm{x}^{\beta}} + \frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\beta\mu}}{\partial \mathrm{x}^{\alpha}} - \frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\mu\alpha}}{\partial \mathrm{x}^{\beta}} - \frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^{\alpha}}\right]$$

$$R_{\alpha\beta\mu\nu} = -\frac{1}{2}\left[\frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\mu\alpha}}{\partial \mathrm{x}^{\beta}} + \frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^{\alpha}} - \frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\nu\alpha}}{\partial \mathrm{x}^{\beta}} - \frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\beta\mu}}{\partial \mathrm{x}^{\alpha}}\right]$$

$$R_{\beta\alpha\mu\nu} = \frac{1}{2}\left[\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\nu\beta}}{\partial \mathrm{x}^{\alpha}} + \frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\alpha\mu}}{\partial \mathrm{x}^{\beta}} - \frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\mu\beta}}{\partial \mathrm{x}^{\alpha}} - \frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\alpha\nu}}{\partial \mathrm{x}^{\beta}}\right]$$

$$\boxed{R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}}$$

If we swap the first and third indices $(\alpha \leftrightarrow \mu)$, and also the second and fourth $(\beta \leftrightarrow \nu)$, we get:

$$R_{\mu\nu\alpha\beta} = \frac{1}{2}\left[\frac{\partial}{\partial x^{\alpha}}\frac{\partial g_{\beta\mu}}{\partial \mathrm{x}^{\nu}} + \frac{\partial}{\partial x^{\beta}}\frac{\partial g_{\nu\alpha}}{\partial \mathrm{x}^{\alpha}} - \frac{\partial}{\partial x^{\beta}}\frac{\partial g_{\alpha\mu}}{\partial \mathrm{x}^{\nu}} - \frac{\partial}{\partial x^{\alpha}}\frac{\partial g_{\nu\beta}}{\partial \mathrm{x}^{\mu}}\right]$$

$$\boxed{R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}}$$

If we cyclically permute the last three indices β, μ and ν and add the three terms, we get:

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = \frac{1}{2}\left[\cancel{\frac{\partial}{\partial x^{\beta}}\frac{\partial g_{\alpha\nu}}{\partial \mathrm{x}^{\mu}}} + \cancel{\frac{\partial}{\partial x^{\alpha}}\frac{\partial g_{\beta\mu}}{\partial \mathrm{x}^{\nu}}} - \frac{\partial}{\partial x^{\beta}}\frac{\partial g_{\alpha\mu}}{\partial \mathrm{x}^{\nu}} - \cancel{\frac{\partial}{\partial x^{\alpha}}\frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^{\mu}}}\right]$$

$$+\frac{1}{2}\left[\frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\alpha\mu}}{\partial \mathrm{x}^{\beta}} + \cancel{\frac{\partial}{\partial x^{\alpha}}\frac{\partial g_{\nu\beta}}{\partial \mathrm{x}^{\mu}}} - \cancel{\frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\alpha\beta}}{\partial \mathrm{x}^{\mu}}} - \cancel{\frac{\partial}{\partial x^{\alpha}}\frac{\partial g_{\mu\nu}}{\partial \mathrm{x}^{\beta}}}\right]$$

$$+\frac{1}{2}\left[\cancel{\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\alpha\beta}}{\partial \mathrm{x}^{\nu}}} + \cancel{\frac{\partial}{\partial x^{\alpha}}\frac{\partial g_{\mu\nu}}{\partial \mathrm{x}^{\beta}}} - \cancel{\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\alpha\nu}}{\partial \mathrm{x}^{\beta}}} - \cancel{\frac{\partial}{\partial x^{\alpha}}\frac{\partial g_{\mu\beta}}{\partial \mathrm{x}^{\nu}}}\right]$$

$$\boxed{\boldsymbol{R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0}}$$

2.11.2 Symmetry properties

From the above expression, we can derive the following symmetries of the Riemann tensor:

1. **Antisymmetry in the last two indices:**

$$R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu}$$

2. **Antisymmetry in the first two indices:**

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}$$

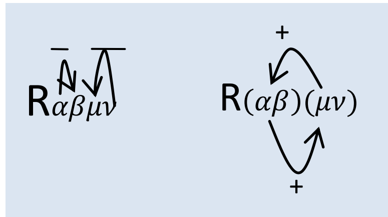
3. **Symmetry under exchange of index pairs:**

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\,\alpha\beta}$$

4. **The first Bianchi identity (cyclic symmetry):**

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0$$

Or expressed as:



The antisymmetry means that the tensor changes sign when these indices are exchanged, which is related to the direction of loop integration in parallel transport.

2.11.3 Number of Independent Components

In a four-dimensional space-time, with four values per index, a random (0, 4) tensor would have 256 components. Due to the symmetries mentioned above, this number is drastically reduced:

- Due to antisymmetry in $(\alpha\beta)$ and $(\mu\nu)$: from $4^4 = 256$ to $\binom{4}{2} \times \binom{4}{2} = 6 \times 6 = 36$
- Symmetry between the pairs: $36 \rightarrow \frac{6\times(6+1)}{2} = 21$
- Bianchi identity: further reduces the number to **20 independent components**

2.11.4 Key insights

- The **Riemann tensor** $R_{\rho\sigma\mu\nu}$ has multiple symmetries, which greatly limits the number of independent components:
    1. **Antisymmetry in the last two indices**:
    $$R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}$$
    2. **Antisymmetry in the first two indices**:
    $$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$$
    3. **Symmetry when swapping index pairs**:
    $$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$$
    4. **Bianchi identity (contraction property)**:
    $$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0$$
- Due to these symmetries, the Riemann tensor in 4D has **only 20 independent components**, not 256.

So although the original expression of the Riemann tensor seems complex, thanks to its rich symmetry structure, it is completely determined by only 20 independent components. These components represent all possible forms of curvature in a four-dimensional space-time and thus form the core of the geometric description of gravity in general relativity.

2.11.5 Intuitive explanation

Imagine a cube with 4 index positions—in theory, there would be 4 × 4 × 4 × 4 = 256 components.

But due to symmetries such as:

- "if you swap these two indices, only the sign changes"
- "if you swap the pairs, it remains the same"

it turns out that **many of those 256 values are related to each other**.

Think of a painting with mirror symmetry: if you know one half, you also know what should be on the other side. The same applies to the structure of the Riemann tensor.

These properties are not coincidental, but arise from the way in which the tensor is derived from the metric and its derivatives.

Table overview:

Symmetry	Explanation
$R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}$	Antisymmetry in last two indices
$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$	Antisymmetry in the first two indices
$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$	Swapping index pairs
Bianchi identity	Linear relationship between permutations of indices
Total in 4D	20 independent components

2.12 Bianchi Identity and Ricci Tensor

The Bianchi identity plays a crucial role in deriving Einstein's field equations. Although the Riemann curvature tensor itself does not appear directly in these equations, we can derive two other important curvature quantities from this tensor via contraction: the Ricci tensor and the Ricci scalar.

In this chapter, we will introduce these three fundamental objects and explain their interrelationships, starting with the derivation of the Bianchi identity.

2.12.1 Bianchi identity

The Bianchi identity is:

$$\nabla_\sigma R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\sigma\mu} + \nabla_\mu R_{\alpha\beta\nu\sigma} = 0$$

From the previous chapter 2.11 Symmetries and independent components we know that at the origin of a Local Inertial frame, the Riemann tensor can be written as:

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left[\frac{\partial}{\partial x^\beta}\frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial}{\partial x^\alpha}\frac{\partial g_{\beta\mu}}{\partial x^\nu} - \frac{\partial}{\partial x^\beta}\frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial}{\partial x^\alpha}\frac{\partial g_{\beta\nu}}{\partial x^\mu}\right]$$

Because the Christoffel symbols disappear at the origin of this system, the covariant derivative there becomes equal to the ordinary derivative:

$$\nabla_\sigma V^\alpha = \frac{\partial V^\alpha}{\partial x^\sigma}$$

So, at the origin, the following applies:

$$\nabla_\sigma R_{\alpha\beta\mu\nu} = \frac{\partial R_{\alpha\beta\mu\nu}}{\partial x^\sigma}$$

Substituting the expression for the Riemann tensor yields:

$$\nabla_\sigma R_{\alpha\beta\mu\nu} = \frac{\partial}{\partial x^\sigma} R_{\alpha\beta\mu\nu} = \frac{1}{2}\left[\frac{\partial}{\partial x^\sigma}\frac{\partial}{\partial x^\beta}\frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial}{\partial x^\sigma}\frac{\partial}{\partial x^\alpha}\frac{\partial g_{\beta\mu}}{\partial x^\nu} - \frac{\partial}{\partial x^\sigma}\frac{\partial}{\partial x^\beta}\frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial}{\partial x^\sigma}\frac{\partial}{\partial x^\alpha}\frac{\partial g_{\beta\nu}}{\partial x^\mu}\right]$$

By cyclically permuting the index of the derivative with the last two indices, μ, ν , of the tensor, we obtain:

$$\nabla_\nu R_{\alpha\beta\sigma\mu} = \frac{\partial}{\partial x^\nu} R_{\alpha\beta\sigma\mu} = \frac{1}{2}\left[\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\beta}\frac{\partial g_{\mu\alpha}}{\partial x^\sigma} + \frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\alpha}\frac{\partial g_{\beta\sigma}}{\partial x^\mu} - \frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\beta}\frac{\partial g_{\alpha\sigma}}{\partial x^\mu} - \frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\nu}\frac{\partial g_{\beta\mu}}{\partial x^\sigma}\right]$$

$$\nabla_\mu R_{\alpha\beta\nu\sigma} = \frac{\partial}{\partial x^\mu} R_{\alpha\beta\nu\sigma} = \frac{1}{2}\left[\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\beta}\frac{\partial\,\partial g_{\sigma\alpha}}{\partial x^\nu} + \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\alpha}\frac{\partial g_{\beta\nu}}{\partial x^\sigma} - \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\beta}\frac{\partial g_{\alpha\nu}}{\partial x^\sigma} - \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\alpha}\frac{\partial g_{\beta\sigma}}{\partial x^\nu}\right]$$

By adding these three equations and using the commutativity of partial derivatives, we see that the terms cancel each other out in pairs, and we obtain ***the Bianchi identity***:

$$\boxed{\boldsymbol{\nabla_\sigma R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\sigma\mu} + \nabla_\mu R_{\alpha\beta\nu\sigma} = 0}}$$

This Bianchi identity is a tensor equation that is universally valid—in every coordinate system.

2.12.2 Key insights

- The **Bianchi identity** is a fundamental identity for the Riemann tensor:

$$\nabla_\lambda R^\rho_{\sigma\mu\nu} + \nabla_\mu R^\rho_{\sigma\nu\lambda} + \nabla_\nu R^\rho_{\sigma\lambda\mu} = 0$$

- Contraction leads to the so-called **contracted Bianchi identity**:

$$\nabla^{\mu}\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) = 0$$

- This contracted version is **crucial for the consistency of the Einstein field equations**.
- It implies that the derivative of the tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is zero:

$$\nabla^{\mu} G_{\mu\nu} = 0$$

- This corresponds to the conservation of energy and momentum in curved space-time.

2.12.3 Intuitive explanation

The Riemann tensor is not just any random object—it must satisfy deeper structural rules. The **Bianchi identity** is one such rule: a kind of **internal consistency** of the curvature of space-time.

When working with vectors, you say: "the divergence of the force is zero if there are no sources."
With tensors, you say something similar: *the structure of the curvature is organized in such a way that certain combinations of it always* disappear—and that means, among other things, **that the Einstein equations do not simply allow energy to appear out of nowhere**.

The contracted Bianchi identity is essential because it guarantees that the **Einstein tensor** $G_{\mu\nu}$ automatically satisfies a conservation law: energy and momentum are conserved in any curved space-time.

Table overview:

Quantity	**Meaning**
Bianchi identity	Structural symmetry of Riemann tensor
Contracted Bianchi identity	Implies $\nabla^{\mu} G_{\mu\nu} = 0$
Einstein tensor $G_{\mu\nu}$	$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$
Physical meaning	Guarantees conservation of energy and momentum in curved space

2.12.4 The Ricci Tensor

In the next chapter, we will deal with the energy-momentum tensor. This tensor is a rank-2 tensor. For this reason, we must adapt the rank-4 Riemann tensor to a rank-2 tensor, which is called the Ricci tensor. This can be done by multiplying the covariant Riemann tensor by a rank-2 contravariant metric tensor, sharing two common indices. This process is called contraction.

By contracting the first and third indices of the Riemann tensor, we obtain the ***Ricci tensor***:

$$g^{\alpha\beta} R_{\alpha\mu\beta\nu} = R^{\beta}_{\mu\beta\nu} = R_{\mu\nu}$$

The Ricci tensor is symmetric:

$$R_{\mu\nu} = R_{\nu\mu}$$

2.12.5 The Ricci Scalar

By multiplying the Ricci tensor by the metric tensor with the same indices, the Ricci tensor is contracted, resulting in the ***Ricci scalar***:

$$R = g^{\mu\nu} R_{\mu\nu}$$

This scalar curvature *R* is the trace of the Ricci tensor.

These tensors—the Ricci tensor and Ricci scalar—together with the metric $g_{\mu\nu}$ form the building blocks of Einstein's field equations. The Bianchi identity also guarantees the conservation laws that follow from these equations.

2.12.6 Key insights

- The **Ricci tensor** $R_{\mu\nu}$ is a *contraction* of the Riemann tensor:

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$$

- It contains information about **how volumes change** in curved space-time (think of the stretching or contraction of geodesic bundles).
- The **Ricci scalar** *R* is a further contraction:

$$R = g^{\mu\nu} R_{\mu\nu}$$

- These quantities are **coordinate-independent** and form the basis of the Einstein field equations.
- Whereas the Riemann tensor fully describes local curvature, the Ricci tensor and scalar are mainly **summary measures of curvature** on a larger scale.

2.12.7 Intuitive explanation

Consider a group of particles in free fall in a small volume. If that volume begins to **shrink or stretch** as time passes, it is due to the **Ricci tensor**.

Where the Riemann tensor tells us *how curvature twists directions*, the Ricci tensor tells us:

- "how does curvature affect the shape of a dust or light beam?"

The **Ricci scalar** can be seen as a summary in a single number of how "curved" space-time is at a given point.

You could say:

- *Riemann* = complete picture of curvature
- *Ricci tensor* = effect on volumes
- *Ricci scalar* = total curvature summarized in a single value

Table overview:

Quantity	**Definition**	**Interpretation**
Riemann tensor	$R^{\rho}_{\sigma\mu\nu}$	Total local curvature
Ricci tensor	$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$	Volume change / summarized curvature
Ricci scalar	$R = g^{\mu\nu} R_{\mu\nu}$	Total curvature in one number

2.13 Energy Impulse Tensor

The ultimate goal of general relativity is to establish a relationship between the geometry of space-time and the matter or energy that distorts it. This requires a suitable mathematical object that describes the content of space-time: the **energy-momentum tensor**.

In the special theory of relativity, it has already been demonstrated that mass, energy, and momentum are interconnected. This relationship is expressed by the well-known energy-momentum equation:

$$|P|^2 = (m_0 c)^2$$

$$|P|^2 = \eta_{\mu\nu} P^\mu P^\nu = \frac{E^2}{c^2} - p_x^2 - p_y^2 - p_z^2 = \frac{E^2}{c^2} - p^2$$

$$=> \ (m_0 c)^2 = \frac{E^2}{c^2} - p^2$$

From which it follows:

$$\boxed{E^2 = p^2 c^2 + {m_0}^2 c^4}$$

This suggests that, within general relativity, **not only mass**, but also **energy and momentum** contribute to the gravitational field.

In the Newtonian limit, Poisson's equation describes the gravitational field Φ , generated by a mass density ρ (see: equation 16 in Appendix 7):

$$-\vec{\nabla}\cdot\vec{g} = -\vec{\nabla}\cdot\left(-\vec{\nabla}\Phi\right) = 4\pi G\rho$$

This raises the question: what is the relativistic equivalent of energy density? Is it a scalar, a vector, or something else?

2.13.1 Transformation properties: the example of a dust cloud

Consider a volume $dx \cdot dy \cdot dz$ filled with non-interacting particles that are at rest relative to each other—a so-called **dust cloud**. In the rest frame ***S*** of this cloud, the energy density is:

$$\boldsymbol{\rho_0 = m_0 n_0}$$

where m_0 is the rest mass of a particle and n_0 is the number density.

In another reference frame $S^{'}$, which moves at a velocity *v* in the x-direction, the Lorentz transformation yields:

- Mass: $m_0 \to m_0\gamma$,
- Density: $n_0 \to n_0\gamma$ (due to length contraction)
- So: $\rho = \rho_0\gamma^2$

Since ρ is **not invariant**, it cannot be a scalar. **Nor** is it **a component of a four-vector**, because then it would only transform linearly with γ. The transformation γ^2 suggests that ρ behaves like a component of a **rank-2 tensor** - namely, like the *tt*-component of a symmetric tensor.

2.13.2 The energy-momentum tensor of matter

The four-velocity vector of the dust cloud in S' is:

$$u^{\mu} = \frac{\partial x^{\mu}}{\partial \tau} = \frac{\partial x^{\mu}}{\partial t}\frac{dt}{\partial \tau} = v^{\mu}\frac{dt}{\partial \tau} = v^{\mu}u^{t}$$

$$u^{\mu} = \gamma(1, v) = \begin{pmatrix} \gamma \\ v_x\gamma \\ v_y\gamma \\ v_z\gamma \end{pmatrix} = \begin{pmatrix} u^t \\ v_x u^t \\ v_y u^t \\ v_z u^t \end{pmatrix}$$

With $u^t = \gamma$, and knowing that the energy of each particle is $p^t = mu^t$, the total energy density is:

$$\rho = np^t = (n_0 u^t)(mu^t) = (n_0 m)u^t u^t = \rho_0 (u^t)^2$$

This suggests that ρ is the *tt*-component of a rank-2 tensor of the form:

$$T^{\mu\nu} = T^{\nu\mu} = \rho_0 u^{\mu}u^{\nu}$$

This tensor is **symmetric** ($T^{\mu\nu} = T^{\nu\mu}$) and is called the **energy-momentum tensor**, also known as the **stress-energy tensor** for matter.

This tensor forms the link between matter/energy and the curvature of space-time in Einstein's field equations. In later chapters, we will see how this tensor appears on the right-hand side of Einstein's equations.

2.13.3 Physical Meaning of the Energy-Momentum Tensor

The energy-momentum tensor is a second-order tensor, which means that it contains 16 components in the form of a 4×4 matrix:

$$T^{\mu\nu} = \begin{pmatrix} T^{tt} & T^{tx} & T^{ty} & T^{tz} \\ T^{xt} & T^{xx} & T^{xy} & T^{xz} \\ T^{yt} & T^{yx} & T^{yy} & T^{yz} \\ T^{zt} & T^{zx} & T^{zy} & T^{zz} \end{pmatrix}$$

As discussed earlier, T^{tt} represents the energy density, or the density of relativistic mass. But what do the other 15 components mean physically?

2.13.4 Time-space components: energy flow

Let's first look at the component T^{tx}. From the definition:

$$T^{tx} = \rho_0 u^t u^x = (n_0 m) u^t u^x = (n_0 u^t)(m u^x) = (n_0 u^t)(m u^t) v_x = n p^t v_x$$

We can rewrite this as:

$$T^{tx} = \frac{n A v_x dt \cdot p^t}{A dt}$$

Here, $A v_x dt$ represents the volume of matter that moves through a surface *A* during the time interval *dt*, perpendicular to the *x*-direction. This volume corresponds to the number of particles that pass through that surface. So:

T^{tx} is the **energy flux** per unit area per unit time in the *x*-direction.

In the same way, represent:

- T^{ty}: the energy flow in the *y*-direction
- T^{tz}: the energy flow in the *z*-direction

Because $T^{\mu\nu}$ is symmetric ($T^{\mu\nu} = T^{\nu\mu}$), the following applies:

$$T^{xt} = T^{tx}, \qquad T^{yt} = T^{ty}, \qquad T^{zt} = T^{tz}$$

2.13.5 Time-space components: impulse flows (stress)

Let us now consider the components with both indices spatially, i.e., T^{kl} with *k, l* ∈ {*x,y,z*}. Then the following applies:

$$T^{kl} = \rho_0 u^k u^l = (n_0 m) u^k u^l = (n_0 m) u^t v_k u^l$$
$$= (n_0 u^t) v_k (m u^l) = n v_k (m u^l) = n v_k p^l$$

Again, we can write this as:

$$T^{kl} = \frac{n A v_k dt \cdot p^l}{A dt}$$

Here, $nAv_k dt$ is the volume flowing in the direction *k* through surface *A*, and so T^{kl} is the **flux of momentum component $\boldsymbol{p^l}$** in the **direction *k***.

For example:

- T^{xz}: flux of z-momentum in the *x*-direction
- T^{xy}: flux of y-momentum in the *x*-direction
- T^{zz}: flux of z-momentum in the *z*-direction (pressure)

Because the tensor is symmetric, the following also applies:

$$T^{xz} = T^{zx}, \qquad T^{xy} = T^{yx}, \qquad T^{yz} = T^{zy}, \dots ..$$

2.13.6 In summary:

- $\boldsymbol{T^{tt}}$ = energy density
- $\boldsymbol{T^{ti}}$ ***or*** $\boldsymbol{T^{it}}$ = energy flow in direction *i*
- $\boldsymbol{T^{ij}}$ = flux of momentum *j* in direction *i* (stress, pressure, and shear stress)

This interpretation makes it clear why $T^{\mu\nu}$ is the correct object to describe the **complete physical content** of a system—from energy and mass density to momentum flows and stresses—and thus acts as a source of gravity in general relativity.

2.13.7 Covariant Differentiation of the Energy-Momentum Tensor

In the flat space-time of Special Relativity, the laws of conservation of energy and momentum—i.e., the fact that energy and momentum are neither created nor destroyed—can be expressed mathematically as:

$$0 = \frac{\partial\, T^{\mu\nu}}{\partial x^{\nu}} = \partial^{\nu}\, T^{\mu\nu} = T^{\mu\nu}_{,\nu}$$

This expression is a direct consequence of **Noether's theorem**, applied to the translation invariance of space and time: the laws of nature do not change when we move the system a little bit in space or time. This symmetry leads to the conservation of momentum and energy.

2.13.8 From Flat to Curved Space-time

In general relativity, we describe physics in curved space-time, in which ordinary derivatives are not sufficient. We therefore replace the partial derivative with the **covariant derivative**:

$$\partial_{\nu} \;\rightarrow\; \nabla_{\nu}$$

Applied to the energy-momentum tensor, this yields:

$$\boxed{0 = \nabla_{\nu}\, T^{\mu\nu} = T^{\mu\nu}_{;\nu}}$$

This equation is a tensor equation and is therefore **generally** covariant—that is, valid in any coordinate system, flat or curved. This makes it a natural candidate for a fundamental conservation principle within general relativity.

2.14 Einstein Tensor

The Poisson equation for the gravitational field in classical (Newtonian) mechanics is as follows (see equation_appendix_5_16):

$$-\vec{\nabla}\cdot\vec{g} = -\vec{\nabla}\cdot\left(-\vec{\nabla}\Phi\right) = 4\pi G\rho$$

Where Φ is the gravitational potential, and ρ is the mass density.

Our goal now is to find a relativistic generalization of this equation. As we saw in section2.13.3, the classical mass density ρ is replaced in general relativity by the energy-momentum tensor $T^{\mu\nu}$. This tensor describes not only mass, but also energy, momentum, and pressure—all forms of energy content of space-time.

It is then obvious to assume that Einstein's relativistic field equation must have the following form:

$$G^{\mu\nu} = \kappa \, T^{\mu\nu}$$

Here, $G^{\mu\nu}$ is the **Einstein tensor** and κ is a constant yet to be determined. The Einstein tensor contains all information about the curvature of space-time and fulfills the role of the left-hand side of the field equation.

2.14.1 Requirements for the Einstein tensor

Based on the physical and mathematical requirements that the field equation must satisfy, the Einstein tensor $G^{\mu\nu}$ must have the following properties:

- It must **vanish in flat space-time**, as $\vec{g} = 0$ in the absence of mass.
- It must **describe** the **space-time curvature** in a way that **is linearly dependent on the Riemann curvature tensor**.
- It must be a **symmetric rank-2 tensor**, just like $T^{\mu\nu}$.
- It must have a **zero-point divergence**: $\nabla_\nu G^{\mu\nu} = 0$, so that the law of conservation of energy and momentum is preserved$(\nabla_\nu \, T^{\mu\nu} = 0)$.
- In the **Newtonian limit,** it must reduce to the Poisson equation: $\nabla^2 \Phi = 4\pi G\rho$.

In the next chapter, we will derive the concrete form of the Einstein tensor that satisfies all these conditions.

2.14.2 First Attempt with the Ricci Tensor as Solution

As we saw in chapter2.8, the gravitational potential Φ is linked to the 00 component of the metric via:

$$\boxed{\frac{d^2\vec{r}}{dt^2} = -\vec{\nabla}\Phi = -\overrightarrow{grad}\Phi \quad \text{met} \quad \Phi = \frac{c^2 h_{00}}{2}}$$

It seems logical to look for a tensor that, like the Laplacian, contains second derivatives of the metric. The Riemann tensor satisfies this condition and is also the only known tensor that fundamentally describes the curvature of space-time.

Because we need a rank-2 tensor (as required in the Einstein field equation), it makes sense to first look at the contracted form of the Riemann tensor: the **Ricci tensor**. We recall:

$$\boxed{R^{\alpha}_{\mu\sigma\nu} = \left(\frac{d\Gamma^{\alpha}_{\mu\nu}}{dx^{\sigma}} - \frac{d\Gamma^{\alpha}_{\mu\sigma}}{dx^{\nu}} + \Gamma^{\alpha}_{\sigma\gamma}\,\Gamma^{\gamma}_{\mu\nu} - \Gamma^{\alpha}_{\nu\gamma}\,\Gamma^{\gamma}_{\mu\sigma} \right)}$$

By contracting the top and third indices, we obtain the Ricci tensor:

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$$

$$\boxed{R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = \left(\frac{d\Gamma^{\alpha}_{\mu\nu}}{dx^{\alpha}} - \frac{d\Gamma^{\alpha}_{\mu\alpha}}{dx^{\nu}} + \Gamma^{\alpha}_{\alpha\gamma}\Gamma^{\gamma}_{\mu\nu} - \Gamma^{\alpha}_{\nu\gamma}\Gamma^{\gamma}_{\mu\alpha}\right)}$$

In the Newtonian limit, for a weak and static gravitational field, only one term contributes to R_{00} . We find:

$$R_{00} = R^{\alpha}_{00\alpha} = \Gamma^{\alpha}_{00,\alpha} - \Gamma^{\alpha}_{0\alpha,0} + \mathcal{O}(h^2) = \Gamma^{i}_{00,i}$$

Since we are limiting ourselves to a static field, the time derivative disappears and we are left with:

$$R_{00} = \Gamma^{i}_{00,i}$$

Using the previously derived result for the Christoffel symbol in this approximation:

$$\Gamma^{i}_{00} = -\frac{1}{2}\mathrm{g}^{\mathrm{ij}} g_{00,j} \approx \frac{1}{2}\partial_i h_{00}$$

With the approximation $g^{ij} = \eta^{ij}$ and $g_{00,j} = h_{00,j}$, we obtain:

$$\Gamma^{i}_{00} = -\frac{1}{2}\eta^{ij} h_{00,j} = \frac{1}{2}\delta^{i}_{j} h_{00,j}$$

$$\Gamma^{i}_{00,i} = \frac{1}{2}\delta^{i}_{j} h_{00,ij} = \frac{1}{2} h_{00,ii}$$

$$R_{00} = \Gamma^{i}_{00,i} = \frac{1}{2}(\partial_1^2 h_{00} + \partial_2^2 h_{00} + \partial_3^2 h_{00})$$

Substituting $h_{00} = 2\Phi/\mathrm{c}^2$ gives:

$$R_{00} = \frac{1}{2}\nabla^2 h_{00} = \frac{1}{\mathrm{c}^2}\nabla^2\Phi$$

And so:

$$R_{00} = \frac{4\pi G\rho}{\mathrm{c}^2}$$

This result suggests that a field equation of the form:

$$R_{\mu\nu} = \kappa T_{\mu\nu}$$

could satisfy the Newtonian limit, with $\kappa = 8\pi G/c4$ as the candidate constant.

Einstein was indeed initially convinced of this equation in 1915. With it, he even solved the long-standing problem of the precession of Mercury's perihelion. In a letter, he wrote enthusiastically:

"For a few days, I was beside myself with joyful excitement."

However, he ultimately had to reject this first attempt. The reason was that the Ricci tensor generally does **not** have **a zero divergence**, whereas the energy-momentum tensor $T_{\mu\nu}$ does $\left(\nabla^{\nu} T_{\mu\nu} = 0\right)$. As a result, this form could not satisfy the required conservation of energy and momentum.

2.14.3 Second Attempt

There is a tensor that is closely related to the Ricci tensor and that is suitable as the left-hand side of the Einstein field equations: the **Einstein tensor**. This is defined as:

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}$$

Here, $R = R_a^a$ the **Ricci scalar**, or scalar curvature.

This tensor already satisfies several requirements:

- It is **symmetrical**, as required by the symmetry of $T^{\mu\nu}$;
- It is of **rank 2**;
- It describes the **space-time curvature**, since it is directly constructed from the Ricci tensor and thus indirectly from the Riemann tensor.

What remains to be shown is that the **covariant divergence** of the Einstein tensor is zero:

$$\nabla_\nu G^{\mu\nu} = 0$$

This is essential, because only then can it be consistently linked to the energy-momentum tensor $T^{\mu\nu}$, for which the following also applies $\nabla_\nu T^{\mu\nu} = 0$ (see section2.13.2). (See also section2.5.2, equation 15).

We derive this result using the **Bianchi identity**, which reads:

$$\nabla_\sigma R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\sigma\mu} + \nabla_\mu R_{\alpha\beta\nu\sigma} = 0$$

We multiply this identity by the metric terms $g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu}$. Because the derivatives of the metric in a local inertial frame are zero, these factors may be moved inside:

$$\nabla_\sigma\left(g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\alpha\beta\mu\nu}\right) + \nabla_\nu\left(g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\alpha\beta\sigma\mu}\right) + \nabla_\mu\left(g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\alpha\beta\nu\sigma}\right) = 0$$

$$\nabla_\sigma(g^{\gamma\sigma} R) + \nabla_\nu\left(g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\alpha\beta\sigma\mu}\right) + \nabla_\mu\left(g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\alpha\beta\nu\sigma}\right) = 0$$

$$\nabla_\sigma(g^{\gamma\sigma} R) + \nabla_\nu\left(g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\sigma\mu\alpha\beta}\right) + \nabla_\mu\left(g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\nu\sigma\alpha\beta}\right) = 0$$

$$\nabla_\sigma(g^{\gamma\sigma} R) - \nabla_\nu\left(g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\mu\sigma\alpha\beta}\right) - \nabla_\mu\left(g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\nu\sigma\beta\alpha}\right) = 0$$

Using the definition of the Ricci tensor $R^{\mu\nu} = g^{\mu\beta} g^{\nu\sigma} R_{\beta\sigma}$ (step 3) and renaming the indices (step 4), we obtain:

$$\nabla_\sigma(g^{\gamma\sigma} R) - \nabla_\nu\left(g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\mu\sigma\alpha\beta}\right) - \nabla_\mu\left(g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\nu\sigma\beta\alpha}\right) = 0$$

$$\nabla_\sigma(g^{\gamma\sigma} R) - \nabla_\nu\left(g^{\gamma\sigma} g^{\beta\nu} R_{\sigma\beta}\right) - \nabla_\mu\left(g^{\gamma\sigma} g^{\alpha\mu} R_{\sigma\alpha}\right) = 0$$

$$\nabla_\sigma(g^{\gamma\sigma} R) - \nabla_\nu(R^{\gamma\nu}) - \nabla_\mu(R^{\gamma\mu}) = 0$$

$$\nabla_\sigma(g^{\gamma\sigma} R) - \nabla_\sigma(R^{\gamma\sigma}) - \nabla_\sigma(R^{\gamma\sigma}) = 0$$

$$\nabla_\sigma(g^{\gamma\sigma} R) - 2\nabla_\sigma(R^{\gamma\sigma}) = 0$$

$$\nabla_\sigma(2R^{\gamma\sigma} - g^{\gamma\sigma} R) = 0$$

Or rewritten:

$$\nabla_\sigma \left(R^{\gamma\sigma} - \frac{1}{2} g^{\gamma\sigma} R \right) = 0$$

And thus:

$$\nabla_\nu G^{\mu\nu} = 0$$

2.14.4 Conclusion

The Einstein tensor $G^{\mu\nu}$ is the correct choice for the left-hand side of the field equation. It is symmetric, constructed from the space-time curvature, and satisfies the conservation of energy and momentum due to its zero divergence. This makes the equation:

$$G^{\mu\nu} = \kappa T^{\mu\nu}$$

a solid candidate for the general relativistic generalization of the laws of gravity.

2.15 Einstein Field Equations

In the previous two chapters, we derived the two quantities that form the core of the field equations in general relativity:

- The **Einstein tensor** $G^{\mu\nu}$, which describes the space-time curvature, and
- The **energy-momentum tensor**$T^{\mu\nu}$, which represents the matter-energy content of space-time.

These two quantities are linked in the form:

$$G^{\mu\nu} = \kappa T^{\mu\nu}$$

where κ is a constant yet to be determined.

2.15.1 Goal: restoration of Newton in the weak field limit

To find the value of κ , we require that this equation reduces to Newton's classical law of gravity in the **Newtonian limit** (weak, static fields and low speeds). This ensures that general relativity is consistent with classical theories in their domain of application.

2.15.2 Alternative formulation of the field equation

Einstein also wrote the field equations in an alternative, equivalent form. This reads:

$$\boldsymbol{G_{im} = -\chi\left(T_{im} - \frac{1}{2} g_{im} T\right),} \qquad \boldsymbol{(2a)}$$

where:

- χ is a constant (related to κ),
- $T = T_\sigma^\sigma$ is the trace **tensor** of $T_{\mu\nu}$, or the contraction of the tensor,
- and the right-hand side as a whole again forms a tensor of rank 2.

Einstein used this formulation in his famous article *"Die Feldgleichungen der Gravitation" (The Field Equations of Gravitation)*, submitted on November 25, 1915, to the Königlich Preußische Akademie der Wissenschaften (Royal Prussian Academy of Sciences). In it, he writes:

"If 'matter' is present in the space under consideration, its energy tensor appears on the right-hand side of (2) [...]. We set

$$\boldsymbol{G_{im} = -\chi\left(T_{im} - \frac{1}{2} g_{im} T\right)}$$

T is the scalar of the energy tensor of 'matter', the right side of (2) is a tensor."

2.15.2.1 In summary

The complete form of Einstein's field equations is therefore:

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} = \kappa T_{\mu\nu}$$

Or equivalently:

$$G_{\mu\nu} = -\chi\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$$

In the next section, we will determine the constant κ by applying the equation to the Newtonian limit. This will allow us to make the connection with the classical law of gravity and thus establish the general theory of relativity in its final form.

2.15.2.2 The Alternative Form of Einstein's Equation

We start from the standard form of the field equation:

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \kappa T^{\mu\nu}$$

By multiplying both sides of this equation by $g_{\mu\nu}$, we obtain:

$$g_{\mu\nu} R^{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\mu\nu} R = \kappa g_{\mu\nu} T^{\mu\nu}$$

According to the definitions of contraction, the following applies:

$$g_{\mu\nu} R^{\mu\nu} = R \quad \text{and} \quad g_{\mu\nu} T^{\mu\nu} = T$$

This makes the equation:

$$R - \frac{1}{2} R \cdot g_{\mu\nu} g^{\mu\nu} = \kappa T$$

Since $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, their product is the Kronecker delta δ^{μ}_{ν}. By contracting this tensor (i.e., taking the sum over the diagonal elements), we obtain:

$$g_{\mu\nu} g^{\mu\nu} = \delta^{\nu}_{\nu} = 1 + 1 + 1 + 1 = 4$$

The equation then reduces to:

$$R - \frac{1}{2} R \times 4 = \kappa T$$

$$R - 2R = \kappa T$$

$$R = -\kappa T$$

We can now substitute this expression for *R* into the original Einstein equation:

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \times (-\kappa T) = \kappa T^{\mu\nu}$$

Which leads to:

$$R^{\mu\nu} + \frac{1}{2} \kappa g^{\mu\nu} T = \kappa T^{\mu\nu}$$

$$\Longrightarrow R^{\mu\nu} = \kappa \left(T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T \right)$$

We can then rewrite this by multiplying both sides by $g_{\alpha\mu} g_{\beta\nu}$:

$$g_{\alpha\mu} g_{\beta\nu} R^{\mu\nu} = g_{\alpha\mu} g_{\beta\nu} \kappa \left(T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T \right)$$

$$\Longrightarrow R_{\alpha\beta} = \kappa \left(T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T \right)$$

If we replace the indices with $\mu\nu$, we get the alternative form:

$$R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$$

$$\Longrightarrow R_{\mu\nu} = \kappa T_{\mu\nu} - \frac{1}{2} \kappa g_{\mu\nu} T$$

Given that we previously saw $R = -\kappa T$, we can also write this as:

$$R_{\mu\nu} = \kappa T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R$$

Which results in:

$$\boldsymbol{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}}$$

2.15.2.3 Conclusion

This derivation confirms the consistency of Einstein's field equation and its equivalence with the alternative formulation proposed in his original publication. Both forms lead to the same physical predictions, but the alternative notation is often used because of its symmetry and simplicity in applications.

2.15.3 Newtonian Limit

In the previous chapter, we saw that in the limit of weak fields and low speeds, the R_{00} component of the Riemann tensor can be approximated as:

$$R_{00} \approx \frac{1}{c^2} \nabla^2 \Phi$$

Furthermore, when the metric $g_{\mu\nu}$ is reduced to the Minkowski metric $\eta_{\mu\nu}$ of flat space-time, we can approximate the Ricci tensor component as follows:

$$R^{\mu\nu} \equiv g^{0\mu} g^{0\nu} R_{\mu\nu} \approx \eta^{0\mu} \eta^{0\nu} R_{\mu\nu} = (-1)(-1) R_{00} = R_{00}$$

Combined, this gives:

$$R_{00} \approx \frac{1}{c^2} \nabla^2 \Phi = \frac{4\pi G \rho}{c^2}$$

In this Newtonian limit, the only non-negligible component of the energy-momentum tensor $T^{\mu\nu}$ is the component $T^{00} = \rho c^2$. This follows from the expression:

$$T^{\mu\nu} = \rho u^\mu u^\nu \quad \text{with} \quad u^i \ll u^0 = c$$

We can then approximate the traced tensor as:

$$T = g_{\mu\nu} T^{\mu\nu} \approx g_{00} T^{00} \approx \eta_{00} T^{00} = T^{00} = \rho c^2$$

We now apply the *00*-component of the Einstein equation:

$$R_{00} = \kappa \left(T_{00} - \frac{1}{2} \eta_{00} T \right)$$

Filling in gives:

$$\frac{4\pi G \rho}{c^2} = \kappa \left(\rho c^2 - \frac{1}{2} \cdot 1 \cdot \rho c^2 \right)$$
$$\Longrightarrow \frac{4\pi G \rho}{c^2} = \frac{1}{2} \kappa \rho c^2$$

From this it follows:

$$\kappa = \frac{8\pi G}{c^4}$$

We can now formulate the Einstein field equations in their standard and alternative forms:

$$\boxed{R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \frac{8\pi G}{c^4} T^{\mu\nu}}$$

Or:

$$\boxed{R^{\mu\nu} = \frac{8\pi G}{c^4}\left(T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T\right)}$$

And in lowered index notation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Or:

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$$

2.15.3.1 Note 1:

The constant $\kappa = \frac{8\pi G}{c^4}$ has an extremely small value:

$$\kappa = \frac{8\pi G}{c^4} \approx 2.071 \times 10^{-43} s^2 m^{-1} kg^{-1}$$

This means that space-time is extremely 'rigid': only enormous amounts of mass or energy cause noticeable curvature.

2.15.3.2 Note 2:

Despite the relatively simple appearance of Einstein's equations, they are in fact extremely complex. For a given distribution of matter and energy (in the form of $T^{\mu\nu}$, the equations form a system of ten coupled, nonlinear, second-order partial differential equations for the metric $g^{\mu\nu}$ These ten equations correspond to the ten independent components of the symmetric metric.

2.15.3.3 Note 3:

The nonlinearity of Einstein's equations has a profound physical significance. It reflects the self-referential nature of gravity: space-time influences matter and energy, but is also influenced by that same matter and energy. As Kevin Brown notes in *Reflection on Relativity*:

"The self-referential nature of the metric field equations is also reflected in their non-linearity. This is inevitable for a theory in which the metric relations between entities determine their 'positions', and those positions in turn influence the metric."

The non-linearity also implies the possibility of interaction between gravitational fields themselves (such as via graviton exchange), which is not possible for photons in the linear Maxwell formalism of electromagnetism, for example.

2.15.3.4 Note 4:

The Einstein equations impose only six independent constraints on the ten components of the metric $g^{\mu\nu}$. The remaining four degrees of freedom are related to the freedom to choose coordinates: we can specify four arbitrary functions via the coordinates $x^{\alpha}(P)$. This overdetermination is a direct consequence of the fact that the Einstein tensor $G^{\mu\nu}$ has a zero-point divergence: $\nabla_{\mu} G^{\mu\nu} = 0$.

2.15.4 Key insights

- The **Einstein field equations** link **space-time curvature** to **energy-momentum content**:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}$$

- The left-hand side, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}$, is the **Einstein tensor**, which encodes the geometry.
- The right-hand side contains the **energy-momentum tensor** $T_{\mu\nu}$, which describes mass, energy, pressure, and flows.
- The constant κ is derived by matching the equations to Newton's law of gravity in the weak-field limit:

$$\kappa = \frac{8\pi G}{c^4}$$

- An alternative, completely equivalent formulation of the field equation is:

$$R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$$

where $T = g^{\mu\nu} T_{\mu\nu}$ is the **trace** of the energy-momentum tensor.

- Contracting both sides of the standard form yields $R = -\kappa T$, which is consistent with the alternative formulation.

2.15.5 Intuitive explanation

Imagine space-time as a flexible but rigid **four-dimensional fabric**. The Einstein equations describe how that fabric **is distorted by the presence of mass and energy**.

Just like a mattress that dents under a heavy ball, space-time deforms around masses. But instead of a push or force, that deformation is **a geometric effect** that determines how objects move—even when they fall "freely."

The equation $G_{\mu\nu} = \kappa T_{\mu\nu}$ then states:

- **What is present in space-time** (matter, energy, radiation)

- **determines how space-time itself looks** (curves, stretches, twists).

For weak fields and low speeds, this automatically leads to Newton's classical gravitational equation—a crucial test for any relativistic theory.

2.15.6 Table: Important quantities in Einstein's field equations

Quantity	Meaning / Role
$R_{\mu\nu}$	Ricci tensor: summarized curvature
R	Ricci scalar: total curvature
$g_{\mu\nu}$	Metric: measurement structure of space-time
$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$	Einstein tensor: measures geometric deformation
$T_{\mu\nu}$	Energy-momentum tensor: distribution of energy and matter
$T = g^{\mu\nu} T_{\mu\nu}$	$T_{\mu\nu}$'s trace: scalar energy density
$\kappa = \frac{8\pi G}{c^4}$	Coupling constant between geometry and physics

2.16 Summary of the Final Formula for General Relativity

In the previous chapters, we discussed the derivation of the Einstein field equations (EFE) step by step. We covered all the necessary building blocks, such as the Riemann tensor, the Ricci tensor, the Ricci scalar, the energy-momentum tensor, and the use of covariant derivatives. In this concluding chapter, we summarize the final result and explain its physical significance.

2.16.1 Einstein's fundamental insight

Einstein's central idea was that gravity is not a force in the classical sense, but the result of the curvature of space-time. This curvature is caused by the presence of mass and energy. His goal was to find a mathematical formula that describes this relationship: how mass and energy influence the geometry of space-time.

The general form of the field equation

Without going through the entire derivation again, we present here the final result of Einstein's theory:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{4}$$

The term $\lambda g_{\mu\nu}$ contains the so-called **cosmological constant (**$\lambda = 1.1056 \times 10^{-52} m^{-2}$), which only becomes noticeable at cosmological scales. For most applications in astrophysics and classical relativity, we can ignore this term, so that the equation simplifies to:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{5}$$

The **left side** of this equation describes the geometry (the curvature) of space-time, while the **right side** represents the content of space (mass, energy, and momentum).

In this equation, ***c*** stands for the speed of light ($2.99792458 * 10^8$ m/s) and ***G*** is the well-known gravitational constant $6.674 * 10^{-11}$ $m^3kg^{-1}s^{-2}$.

2.16.2 Vacuum: outside a mass

In an area without mass or energy, $T_{\mu\nu} = 0$ applies. The field equation then reduces to:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{6}$$

As discussed in section2.15.2.2The Alternative Form of Einstein's Equationalso applies in that case:

$$R = -\frac{8\pi G}{c^4} T = 0 \quad \Longrightarrow \quad R = 0$$

So that remains:

$$R_{\mu\nu} = 0 \tag{7}$$

These are Einstein's so-called vacuum equations.

2.16.3 Explanation of the objects used

The indices μ en ν run from 0 to 3 and refer to the four dimensions of space-time: time (0) and space (1 = x, 2 = y, 3 = z).
Equation (5) therefore contains 16 component equations:

$$\begin{array}{llll}
R_{00} - \frac{1}{2} g_{00} R = \frac{8\pi G}{c^4} T_{00}\,, & R_{01} - \frac{1}{2} g_{01} R = \frac{8\pi G}{c^4} T_{01}, & R_{02} - \frac{1}{2} g_{02} R = \frac{8\pi G}{c^4} T_{02}\,, & R_{03} - \frac{1}{2} g_{03} R = \frac{8\pi G}{c^4} T_{03} \\
R_{10} - \frac{1}{2} g_{10} R = \frac{8\pi G}{c^4} T_{10}\,, & R_{11} - \frac{1}{2} g_{11} R = \frac{8\pi G}{c^4} T_{11}, & R_{12} - \frac{1}{2} g_{12} R = \frac{8\pi G}{c^4} T_{12}\,, & R_{13} - \frac{1}{2} g_{13} R = \frac{8\pi G}{c^4} T_{13} \\
R_{20} - \frac{1}{2} g_{20} R = \frac{8\pi G}{c^4} T_{20}\,, & R_{21} - \frac{1}{2} g_{21} R = \frac{8\pi G}{c^4} T_{21}, & R_{22} - \frac{1}{2} g_{22} R = \frac{8\pi G}{c^4} T_{22}\,, & R_{23} - \frac{1}{2} g_{23} R = \frac{8\pi G}{c^4} T_{23} \\
R_{30} - \frac{1}{2} g_{30} R = \frac{8\pi G}{c^4} T_{30}\,, & R_{31} - \frac{1}{2} g_{31} R = \frac{8\pi G}{c^4} T_{31}, & R_{32} - \frac{1}{2} g_{32} R = \frac{8\pi G}{c^4} T_{32}\,, & R_{33} - \frac{1}{2} g_{33} R = \frac{8\pi G}{c^4} T_{33}
\end{array}$$

Due to symmetry (namely $R_{\mu\nu} = R_{\nu\mu}$), only 10 are independent.

The Ricci tensor $\boldsymbol{R_{\mu\nu}}$ is often written in matrix form as:

$$\boldsymbol{R_{\mu\nu}} = \begin{vmatrix} R_{00} & R_{01} & R_{02} & R_{03} \\ R_{10} & R_{11} & R_{12} & R_{13} \\ R_{20} & R_{21} & R_{22} & R_{23} \\ R_{30} & R_{31} & R_{32} & R_{33} \end{vmatrix}$$

The **metric tensor** $g_{\mu\nu}$, which contains the geometric structure of space-time, also has 10 independent components and completely determines the space-time geometry:

$$\boldsymbol{g_{\mu\nu}} = \begin{vmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{vmatrix}$$

The Ricci scalar ***R*** follows from the contraction of the Ricci tensor with the inverse metric:

$$\boldsymbol{R} = \mathrm{g}^{\mu\nu} R_{\mu\nu}$$

All elements on the left-hand side of equation (5) describe the geography of the space-time under consideration. On the right-hand side**,** we find the **energy-momentum tensor** $T_{\mu\nu}$ **,** which contains all information about matter and energy in the system:

$$\boldsymbol{T_{\mu\nu}} = \begin{vmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{vmatrix}$$

Here, T_{00} stands for the energy density, T_{0i} for the energy flux, and T_{ij} for the momentum flux and pressure components.

2.16.4 Determination of $R_{\mu\nu}$

The Ricci tensor is calculated via contraction of the Riemann tensor:

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu}$$

$$\mathrm{R}_{\mu\nu} = \mathrm{R}^{\rho}_{\mu\rho\nu} = \frac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\rho}} - \frac{\partial \Gamma^{\rho}_{\rho\mu}}{\partial x^{\nu}} + \Gamma^{\rho}_{\rho\lambda}\,\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\,\Gamma^{\lambda}_{\rho\mu} \quad \text{(note 1)}$$

This tensor depends on the **Christoffel symbols**, which themselves consist of derivatives of the metric:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\} \quad \text{(note 1)}$$

This shows that the entire geometry (and therefore also gravity) depends on the metric $g_{\mu\nu}$ and its derivatives.

2.16.5 The Schwarzschild solution

In 1915, Karl Schwarzschild found an exact solution to the field equations in vacuum around a spherically symmetric mass. This led to the well-known **Schwarzschild metric** (see chapter3):

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\emptyset^2$$

This metric applies outside the mass, i.e., in a region where $T_{\mu\nu} = 0$, and therefore:

$$R_{\mu\nu} = 0$$

The Schwarzschild solution is particularly important because it makes experimentally verifiable predictions, such as the deflection of light and the perihelion precession of Mercury.

The metric tensor then consists of the elements:

$$g_{00} = \left(1 - \frac{2GM}{c^2 r}\right), \qquad g_{11} = -\left(1 - \frac{2GM}{c^2 r}\right)^{-1}, \qquad g_{22} = -r^2, \qquad g_{33} = -r^2 \sin^2\theta$$

This is the so-called *trace* of the tensor. Or in tensor form:

$$g_{\mu\nu} = \begin{vmatrix} \left(1 - \frac{2GM}{c^2 r}\right) & 0 & 0 & 0 \\ 0 & -\left(1 - \frac{2GM}{c^2 r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{vmatrix}$$

So because the Schwarzschild equation is used outside a mass, the right-hand side of the Einstein field equations becomes zero ($T_{\mu\nu} = 0$). This causes the field equations to become equation (6), and because R is derived from $R_{\mu\nu}$, equation (6) can only be zero when $R_{\mu\nu} = 0$. So the only relevant equation is $R_{\mu\nu} = 0$. As mentioned earlier, the tensor $R_{\mu\nu}$ is constructed from Christoffel symbols and their derivatives. We have derived and summarized all relevant Christoffel symbols for this metric in Appendix 1.2

The Schwarzschild equation uses the polar or spherical coordinate system to describe the entire space-time; however, due to conservation of angular momentum, physical motion takes place in a single plane. By choosing the appropriate polar coordinate system, this plane can be rotated so that the equatorial plane coincides with the surface under investigation. In that case, the angle $\theta = \pi/2$, where the metric tensor is further simplified to:

$$g_{\mu\nu} = \begin{vmatrix} \left(1 - \frac{2GM}{c^2 r}\right) & 0 & 0 & 0 \\ 0 & -\left(1 - \frac{2GM}{c^2 r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \end{vmatrix}$$

(See also section7.3"Answer to questions concerning Schwarzschild)

2.16.5.1 Note 1

In his document, Einstein uses the opposite sign for the Christoffel symbol $\Gamma^{\rho}_{\mu\nu}$, and the Ricci tensor $R_{\mu\nu}$ also has the opposite sign for the third and fourth terms on the right-hand side of the equation. For the metric, we have used the so-called (+ - - -) notation, also known as the West Coast convention.

2.16.5.2 Final remark

The Einstein field equations form a powerful system of 10 coupled, nonlinear, partial differential equations. Although they can be written compactly, they are rich and complex in content.

They form the starting point for the search for solutions (such as the Schwarzschild solution, cosmological models) and explain a wide range of physical phenomena - from the orbit of Mercury to the expansion of the universe.

As is often said:

"Mass and energy determine the curvature of space-time, and the curvature of space-time determines the movement of mass and energy."

2.16.6 Key insights

- Einstein's central insight: **gravity is not a force**, but the result of **the curvature of space-time** caused by **mass and energy**.
- **Einstein's field equations** form the foundation of general relativity:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

For most practical applications, the cosmological constant $\lambda \approx 1.1 \text{ x } 10^{-52}\ m^{-2}$ is ignored. The equation then reduces to:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- In a vacuum (outside of matter): $T_{\mu\nu} = 0$, so:

  $$R_{\mu\nu} = 0$$

  these are the **vacuum equations**, which lead, among other things, to the **Schwarzschild solution**
- Each term on the **left-hand side** is purely geometric (derived from the metric $g_{\mu\nu}$);
  the **right-hand side** contains physical information (energy, mass, pressure).

2.16.7 Intuitive explanation

Imagine a **four-dimensional elastic fabric**. Matter and energy pull on that fabric and cause it to deform. That deformation determines how objects **move—they follow the curvature** of space-time.

The equation says:

- Left: *"How is space-time curved?"*
- Right: *"What is in space-time that causes this curvature?"*

For example:

- A planet does not move because it is "pulled" by a force,
- but because it **follows** a **geodesic in curved space-time**.

The equations are elegant and powerful:

- They apply everywhere (due to tensor formality),
- reduce to Newton's gravity in the appropriate limit case,
- and predict phenomena such as gravitational waves, black holes, and the expansion of the universe.

Table: Structure of the final equation

Term	**Meaning**
$G_{\mu\nu}$	Geometric side: curvature
$T_{\mu\nu}$	Physical side: energy content
$\nabla^{\mu} G_{\mu\nu} = 0$	Structural conservation principle
$\frac{8\pi G}{c^4}$	Scale factor connecting geometry and physics

These equations form the final piece of the mathematical backbone of general relativity. From here, it is time to **look for** solutions—for example, the Schwarzschild solution or cosmological models.

Magnitude	**Meaning / Role**
$R_{\mu\nu}$	Ricci tensor: measures local curvature
R	Ricci scalar: total scale of curvature ($R_{\mu\nu}$'s trace)
$g_{\mu\nu}$	Metric: determines the measurement structure of space-time
$\lambda g_{\mu\nu}$	Cosmological constant (especially relevant on a cosmic scale)
$T_{\mu\nu}$	Energy-momentum tensor: distribution of energy and matter
$T_{\mu\nu}$	Energy-momentum tensor: describes matter, energy, pressure, and flow
$\frac{8\pi G}{c^4}$	Coupling constant between geometry and physics

Part III – Physical Interpretations

3 Schwarzschild metric

Working with Einstein's field equations is generally quite complex due to their general and tensorial nature. Fortunately, in 1915 Karl Schwarzschild found an exact solution to these equations in a specific case: that of a stationary, spherically symmetric gravitational field in a vacuum. (See Chapter 6: Verification of whether the Schwarzschild Metric satisfies Einstein's Field Equations

In his theory, Einstein considered all possible distributions of mass and energy. Schwarzschild, on the other hand, limited himself to the situation in a vacuum, i.e., outside of matter, where the energy-momentum tensor is zero $\left(T_{\mu\nu} = 0\right)$. He investigated the effect of a central, non-rotating, spherically symmetric mass on the surrounding space-time—for example, the influence of the sun on passing planets or light rays. (For a more detailed overview, see chapter 55.11and Schwarzschild: *"On the Gravitational Field of a Mass Point According to Einstein's Theory "*)

The metric found by Schwarzschild is:

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\emptyset^2 \tag{8}$$

This metric describes the distance between two events in a spherically symmetric gravitational field in terms of the time coordinate *t*, the radial distance *r*, and the angles θ and ϕ . In an infinitely small local region, we can construct a local inertial frame in which the coordinates $cdt, dr, d\theta \text{ and } d\phi$ behave as linear, orthogonal quantities. The metric coefficients are constant in such a local plane, but generally vary with r and θ .

For further considerations on this solution, see the next chapter.

For the complete original derivation of the Schwarzschild metric, see *Schwarzschild, On the Gravitational Field of a Point-Mass, According to Einstein's Theory*, January 13, 1916, and the review article by Oas.

3.1 Discussions about the Schwarzschild metric

3.1.1 Introduction

The Schwarzschild metric is an exact solution of Einstein's field equations for the case of a spherically symmetric mass in a vacuum. Karl Schwarzschild published this solution in 1915, shortly after Albert Einstein formulated the general theory of relativity. This solution is one of the most important applications of the theory and describes how mass influences the structure of space-time.

3.1.2 The Schwarzschild metric in polar coordinates

The Schwarzschild equation in polar coordinates is:

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\emptyset^2 \tag{1}$$

In this equation:

- G is the gravitational constant,
- M is the mass,
- c is the speed of light,
- $t, r, \theta, \emptyset$ the time and space coordinates.

This metric describes the curved space-time outside a spherically symmetric mass, assuming that no other matter is present (vacuum).

3.1.3 Dimension analysis

At first glance, it seems as if the dimensions of this equation are incorrect. In reality, the coefficients are dimensionless, while the coordinates have a length dimension (in meters or m^2 when squared). The meaning of formula (1) is therefore that:

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) d(c^2 t^2) - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - \frac{r^2}{R_p^2} d\left(R_p^2 . \theta^2\right) - \frac{r^2}{R_p^2} \sin^2\theta d\left(R_p^2 . \emptyset^2\right)$$

with $R_p = 1$ meters. This makes it clear that the coordinates $r, \theta, \emptyset$ are treated dimensionally as lengths, while the corresponding coefficients remain dimensionless.

For practical reasons, the original form of equation (1) is usually used, but it is important to realize that $d\theta$ and $d\emptyset$ are given a length dimension here by multiplication with R_p .

3.1.4 Key insights

- The Schwarzschild solution was found in 1915 by **Karl Schwarzschild** as an **exact solution of the Einstein field equations** in the case of a:
    - spherically symmetric,
    - non-rotating,
    - static mass object in a vacuum $T_{\mu\nu} = 0$
- The metric describes how space and time are curved by a mass M, such as a star or planet, **outside of matter.**
- The metric is:

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\emptyset^2$$

- This metric is represented in **polar coordinates** $(t, r, \theta, \emptyset)$, adapted to the spherical symmetry of the problem
- The characteristic scale is the **Schwarzschild radius**:

$$r_s = \frac{2GM}{c^2}$$

- For $r \to \infty$, the metric approximates the flat Minkowski metric, as required for asymptotic flatness in the absence of mass.
- Near $r = r_s$, effects such as **time dilation, horizon formation**, and extreme curvature occur

3.1.5 Intuitive explanation

Imagine a heavy star in empty space. Instead of a force field as in Newton's theory, Einstein says that this mass **distorts space-time itself**.

The Schwarzschild metric shows **how strong this distortion is at different distances**:

- **Time dilation**: the clock ticks slower closer to the mass → determined by the factor:

$$g_{00} = 1 - \frac{r_s}{r} \;\; ook\ wel\ g_{tt}$$

- **Radial distance distortion**: measuring a distance in the r-direction takes up more "physical space" than you think → determined by g_{11} :

$$g_{11} = \left(1 - \frac{r_s}{r}\right)^{-1} ook\ wel\ g_{rr}$$

- **Angular components** (g_{22}, g_{33}) remain classical: surface of spheres with radius *r*.

"The Schwarzschild solution is therefore not an abstract formula, but a concrete, measurable distortion of space and time—visible in the passage of clocks and the behavior of light and planets."

Table overview

Quantity	**Meaning**
ds^2	Line element: measurable distance between events
g_{00}	Determines time dilation (how time passes in the presence of mass)
g_{11}	Determines distortion of radial distances
$r_s = \frac{2GM}{c^2}$	Schwarzschild radius (possible horizon for black hole)
$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\emptyset^2$	Spherical surface element

3.2 Relation of the Schwarzschild Metric and Noether's Theorem

3.2.1 Introduction

One of the most profound insights in modern physics is the deep connection between the symmetries of physical laws and the conservation laws that follow from them. This connection was established in 1918 by Emmy Noether, whose theorem states that every continuous symmetry of a physical system corresponds to a conserved quantity.

The Schwarzschild solution of Einstein's field equations provides an ideal setting in which to illustrate this principle. It describes the spacetime surrounding a spherically symmetric, non-rotating mass, such as an idealized black hole or a non-rotating star.

3.2.2 The Schwarzschild Metric

The geometry of this spacetime is described by the metric:

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\emptyset^2$$

From this metric (the coefficients) we see immediately that:

- it does not explicitly depend on t (time independence),
- it does not explicitly depend on $\emptyset$ (rotational symmetry),
- it is fully spherically symmetric (invariant under arbitrary spatial rotations).

3.2.3 Symmetries and Killing Vectors

In general relativity, symmetries of the metric are represented by Killing vectors ξ^{μ}, which satisfy the **Killing equation:**

$$\nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu} = 0$$

Here, ∇_{μ} denotes the **covariant derivative** with respect to x^{μ}, which incorporates the curvature of spacetime and therefore differs from the ordinary partial derivative ∂_{μ}. For a vector field we have, for example,

$$\nabla_{\mu}V^{\nu} = \partial_{\mu}V^{\nu} + \Gamma^{\lambda}_{\mu\nu}V^{\lambda}$$

where $\Gamma^{\lambda}_{\mu\nu}$ are the Christoffel symbols.

A **Killing vector field** ξ^{μ} represents a direction in spacetime along which the metric remains unchanged. This is the mathematical way of characterizing a symmetry, such as a time translation or a spatial rotation.

Along the geodesic of a test particle, every Killing vector leads to a conserved quantity:

$$u^{\mu}\xi_{\mu} = constant$$

where u^{μ} is the particle's four-velocity. This is the relativistic formulation of Noether's theorem.

It is important to note, however, that a linear relation between two quantities does not itself imply symmetry. A true symmetry means that the form of the physical laws remains unchanged under a given transformation. According to Noether's theorem, such a symmetry guarantees a conserved quantity. For instance, the Schwarzschild metric's independence of t implies time-translation symmetry, which yields energy conservation. Similarly, its independence of $\emptyset$ implies rotational symmetry, which leads to angular momentum conservation. The conserved quantities often appear in linear relations, but these relations are consequences of the symmetries, not the symmetries themselves.

3.2.4 Application to the Schwarzschild Metric

(a) Time Translation

The Killing vector:

$$\xi^{\mu} = (1,0,0,0)$$

corresponds to invariance under time translations. The associated constant of motion is the energy per unit mass:

$$E = \left(1 - \frac{2GM}{c^2 r}\right)c^2\frac{dt}{d\tau}$$

Here $\frac{dt}{d\tau} \neq 0$, since the particle always moves forward in time. What is conserved is the combination $\boldsymbol{g_{00}\frac{cdt}{d\tau}}$: the coefficient of the derivative, not the coordinate itself.

(b) Rotations

There are three Killing vectors corresponding to the three independent rotational symmetries. A simple example is the azimuthal symmetry:

$$\xi^{\mu} = (0,0,0,1)$$

which yields the conserved angular momentum component

$$L = r^2 \sin^2 \theta \frac{d\emptyset}{d\tau}$$

Together, the three rotational symmetries lead to conservation of the full angular momentum vector. In practice, one often writes only the Killing vector for the azimuthal coordinate Ø, which corresponds to conservation of L_z. But two additional Killing vectors correspond to rotations about the other axes, and all three together imply conservation of the complete vector $\left(L_x, L_y, L_z\right)$. Again, the conserved quantity is the coefficient combining the derivative and the metric component.

3.2.5 Physical Significance

These conserved quantities have direct physical consequences:

- **Energy conservation** determines whether a particle falls inward radially or escapes to infinity.
- **Angular momentum conservation** governs the shape of orbits and explains phenomena such as the perihelion precession of Mercury and the bending of light by gravity.

Thus, spacetime symmetries manifest themselves as measurable astrophysical effects, with conservation laws expressed as the **coefficients of derivatives along the geodesic**.

3.2.6 The Scope of Noether's Theorem in General Relativity

In flat spacetime, energy and angular momentum can be defined universally.
In curved spacetime, the situation is more subtle: a global time symmetry may not exist.

The Schwarzschild solution is stationary and asymptotically flat, so energy and angular momentum conservation are well-defined and applicable.
In contrast, in dynamical cosmological spacetimes (such as the expanding universe), global definitions of conserved energy are often impossible.

3.2.7 7. Conclusion

The Schwarzschild metric demonstrates the remarkable power of Noether's theorem within general relativity. The metric's symmetries, expressed through Killing vectors, give rise to conserved energy and angular momentum. These conserved quantities are not the coordinates themselves but the **coefficients associated with derivatives along the geodesic**.

This insight is central to understanding the motion of particles and light in gravitational fields, and it provides a bridge between the mathematical structure of the theory and the observable physical phenomena it predicts.

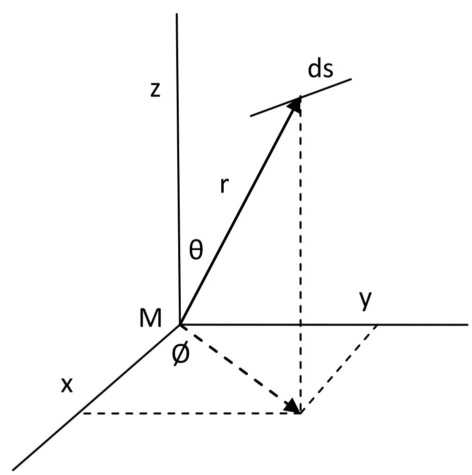
3.3 Physical interpretation of the Schwarzschild metric

Let us now examine the physical meaning of formula (1). Suppose there is an object in space with mass *M*, which we consider to be a point mass. In classical Newtonian mechanics, such a mass causes a gravitational field—a force that acts on other masses in the vicinity.

In general relativity, this idea is fundamentally different: according to Einstein and Schwarzschild, the mass *M* does not generate a force, but distorts the structure of space-time. So there is no longer a force in the classical sense, but a geometric effect.

We choose a coordinate system in which *M* is at the center. When a test particle (with negligible mass) is at rest relative to this mass, it experiences a gravitational force in the Newtonian sense. If we release this particle, it would accelerate towards *M*, just as Newton predicts.

Yet the particle itself does not feel any force. In its own (moving) frame of reference, it experiences nothing special—it simply follows the natural path prescribed by space-time itself. In Einstein's theory, this path is not a straight line, but a geodesic: the shortest or "straightest" path in a curved space-time.



Space coordinates

3.3.1 The chosen coordinate system and the local structure of space

We are working here with a Euclidean coordinate system, which can be understood as a Cartesian system (*t, x, y, z*) or—as in the Schwarzschild solution—a polar system $(t, r, \theta, \emptyset)$. In the polar case, the path followed by a particle depends on all four coordinates. The Schwarzschild metric assigns a coefficient to each differential, which is a function of r and θ but independent of t and $\emptyset$. This reflects the spherical symmetry around the mass M*:* a rotation around the center does not change the physical situation.

It is important to realize that these coordinates are hypothetical: they are defined as if they were located in a flat, non-curved space-time. Schwarzschild found an explicit formula that describes the space-time curvature around a point mass. This formula relates the elementary line segment *ds* (the distance in space-time between two neighboring events) to the chosen coordinate system.

Although space-time is curved, we can consider it to be flat locally—in an infinitesimally small area. Within such a small area*,* we can treat the coordinates *cdt, dr,* $d\theta$*,* and $d\emptyset$ as being mutually perpendicular and straight. The coefficients in the metric can be considered constant there. If we move to another location, these properties remain valid locally, but with modified coefficients as a result of changes in *r* and θ .

By then integrating over *ds* along a path—that is, over all infinitesimal steps together—we obtain the complete trajectory of the particle in curved space-time.

3.3.2 The Schwarzschild metric and the role of proper time

As discussed earlier, the Schwarzschild metric in polar coordinates has the form:

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\emptyset^2 \qquad (1)$$

Here, ds^2 describes the square interval in space-time between two neighboring events.

For greater compactness, we introduce the function:

$$\text{Where: } \sigma = \sqrt{1 - \frac{2GM}{c^2 r}}$$

allowing the metric to be elegantly rewritten as:

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \sigma^{-2} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\emptyset^2 \qquad (1a)$$

Here, $d\tau$ is the **proper time**: the time measured by a clock moving with the object. This is the actual duration experienced by an observer along his own world line.

The coordinate time *dt*, on the other hand, belongs to a hypothetical system in which there is no mass—an ideal "flat" frame of reference. Strictly speaking, *dt* is not directly measurable, except in the limit $r \longrightarrow \infty$, where $\sigma \rightarrow 1$ and space-time becomes flat.

Locally, at a fixed value of *r*, the Schwarzschild metric relates coordinate time and proper time via a simple relationship:

$$\Delta t_{co\ddot{o}rdinate} = \sigma dt$$

where σ depends on the position *r*.

3.3.3 Distance traveled, velocity, and the connection with the Schwarzschild metric

In Schwarzschild space-time, the infinitesimal spatial distanceΔ distance between two events is given by:

$$\Delta distance = \sqrt{\sigma^{-2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\emptyset^2}$$

Where, as mentioned earlier:

$$\sigma = \sqrt{1 - \frac{2GM}{c^2 r}}$$

The corresponding time duration in coordinate time is:

$$\Delta \text{time} = \sigma dt$$

From this, the (local) velocity v of a particle in the frame follows:

$$v^2/c^2 = \frac{1}{c^2}\left(\frac{\Delta distance}{\Delta time}\right)^2 = \frac{(\sigma^{-2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\emptyset^2)}{\sigma^2 c^2 dt^2}$$

This takes into account the curvature of space-time.

If we substitute this expression back into the compact form of the Schwarzschild metric (equation (1a)), we find:

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \frac{\sigma^{-2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\emptyset^2}{\sigma^2 c^2 dt^2} \sigma^2 c^2 dt^2 \qquad (2)$$

which simplifies to:

$$c^2 d\tau^2 = \sigma^2 c^2 dt^2 \left(1 - \frac{\sigma^{-4}}{c^2}\left(\frac{dr}{dt}\right)^2 - \frac{\sigma^{-2} r^2}{c^2}\left(\frac{d\theta}{dt}\right)^2 - \frac{\sigma^{-2} r^2 \sin^2\theta}{c^2}\left(\frac{d\emptyset}{dt}\right)^2\right) = \sigma^2 c^2 dt^2 \left(1 - \frac{v^2}{c^2}\right) \qquad (3)$$

or, written even more compactly:

$$c^2 d\tau^2 = \sigma^2 c^2 dt^2 \left(1 - \frac{v^2}{c^2}\right)$$

where:

$$v^2 = \sigma^{-4}\left(\frac{dr}{dt}\right)^2 + \sigma^{-2} r^2 \left(\frac{d\theta}{dt}\right)^2 + \sigma^{-2} r^2 \sin^2\theta \left(\frac{d\emptyset}{dt}\right)^2 \qquad (3a)$$

This derivation shows how both spatial and temporal curvature together determine the dynamics of a moving particle.

3.3.4 Relationship between proper time and coordinate time *dt*

The previous derivation (equation (3)) directly implies the relationship between proper time $d\tau$ and coordinate time *dt*:

$$d\tau = \frac{\sigma}{\gamma} dt$$

Where:

$$\sigma = \sqrt{1 - \frac{2GM}{c^2 r}} \qquad \text{en} \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{4}$$

Here, σ is a measure of gravitational deceleration (due to mass M) and γ is the Lorentz factor, derived from special relativity.

Because $\gamma \geq 1$ (since $v \leq c$ and $\sigma \leq 1$ (because $r \geq 2GM/c^2$), it follows that:

$$d\tau \leq dt \tag{5}$$

This means that the proper time of the moving object always passes more slowly than the coordinate time in the reference frame.

Since both σ and γ are constant during the interval under consideration (they are only functions of r and v , not of t), we can easily integrate this relationship:

$$\boldsymbol{\tau = \frac{\sigma}{\gamma} t} \tag{5a}$$

where τ is the elapsed proper time and t is the elapsed coordinate time.

Here:

- t is the **coördinate time** of an external observer (for example, someone far from the gravitational field).
- τ is the **proper time** of the particle itself.

And:

- $\frac{dt}{d\tau}$ is the rate at which the coordinate time passes relative to the particle's proper time.
- $\frac{dt}{d\tau}$ is **not a velocity in meters per second** like $\frac{dr}{dt}$, but a relative "time-speed" between the observer's coordinate time and the particle's proper time.
- It does, however, play the same role in the Lagrangian as a kinetic term: the square $\dot{t}^2$ appears in the energy-like conserved quantity.

3.3.5 Behavior of a photon in the Schwarzschild metric

For a photon, the proper time $d\tau$ is zero, because a photon always moves at the speed of light:

$$0 = \sigma^2 c^2 dt^2 - \sigma^{-2} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\emptyset^2 \tag{6}$$

It follows that the spatial distance traveled by the photon is given by:

$$(\Delta distance)^2 = \sigma^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\emptyset^2 \tag{6b}$$

We can now determine the effective speed of light v relative to the chosen coordinate system:

$$c^2 = \left(\frac{\Delta distance}{\Delta time}\right)^2 = \frac{\sigma^{-2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\emptyset^2}{\sigma^2 dt^2} = v^2 \tag{6c}$$

Here, $\Delta time = \sigma dt$, as discussed earlier.

3.3.6 Interpretation:

- In the numerator, we find the 'normal' spatial distance traveled by the photon.
- In the denominator, we see that time is influenced by a factor of σ : the clock at a given location ticks slower due to the influence of gravity.

This shows that, measured in the coordinate time *dt*, the speed of light is effectively **lower** than *c* in the presence of gravity.
In the local (moving) frame, the photon of course still moves at the constant speed *c*.

3.3.7 Alternative description of photon motion

We can also rewrite the relationship between distance traveled and time elapsed in a different way:

$$c^2 = \left(\frac{\Delta distance}{\Delta time}\right)^2 = \frac{\sigma^{-2}(\sigma^{-2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\emptyset^2)}{dt^2} \tag{6d}$$

Here we note:

- The spatial distance is increased by a factor of σ^{-2} (since $\sigma \leq 1$).
- At the same time, the coordinate time *dt* remains unchanged.

3.3.8 Consequence:

Due to this increased distance in the numerator, but an unchanged time in the denominator, the speed of light in the coordinate system appears to be **smaller** than the universal speed of light *c*.

3.3.9 Summary:

From the perspective of the 'universal' coordinate system:

- A photon moves **along a curve** in curved space-time, and

- The effective speed of the photon between two coordinate points (e.g., from A to B) is **less than** *c*.

In comparison:

$$\sigma^2 c^2 = \frac{\sigma^{-2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\emptyset^2}{dt^2} \tag{6e}$$

it can be seen that the speed of light is effectively modified by the factor σ^2.

3.3.10 Behavior

Physically, this means that:

- The intrinsic speed of the photon remains *c* along its world line.
- But the projection of its motion onto the coordinate system appears to be a lower speed due to the curvature of space-time.

In other words:

$$v = \frac{distance}{t} = \frac{distance}{pathlength/c} = \frac{distance}{pathlength}c$$

Because the path length is greater than the "straight-line distance,"$v < c$, as measured in coordinate time.

3.3.11 Relationship between local time on Earth and universal frame time

As noted earlier, coordinate time *dt* is a hypothetical time, defined in a massless environment or at infinite distance $(r = \infty)$. Since we perform our measurements from Earth, we must establish a relationship between (see also section4.6):

- the **proper time** $d\tau_{eart\,h}$ as measured by a clock on Earth, and
- the **coordinate time** *dt* from the universal frame of reference.

As previously derived (see also formula (5a)), the following applies:

$$\boldsymbol{d\tau_{aarde}} = \frac{\boldsymbol{\sigma}_{eart\,h}}{\boldsymbol{\gamma}_{eart\,h}}\boldsymbol{dt}$$

Or, conversely:

$$\boldsymbol{dt} = \frac{\boldsymbol{\gamma}_{eart\,h}}{\boldsymbol{\sigma}_{eart\,h}}\boldsymbol{d\tau}_{eart\,h}$$

where:

- $\boldsymbol{\sigma}_{eart\ h} = \sqrt{1 - \frac{2GM}{c^2 r_{eart\ h}}}$ is the gravitational time dilation factor,
- $\boldsymbol{\gamma}_{eart\ h} = \frac{\mathbf{1}}{\sqrt{1 - \frac{v^2{}_{eart\ h}}{c^2}}}$ is the special relativistic Lorentz factor (due to the rotation of the Earth).

3.3.12 Interpretation:

Time on Earth slows down relatively due to two effects:

1. **gravity** (gravitational time dilation, via σ),
2. the **movement** of the Earth (special relativity, via γ).

For an observer moving with the Earth, his own time $d\tau_{eart\ h}$ naturally continues to run normally - every second remains a second.
However, relative to the universal frame time *dt,* the local seconds run **slightly slower**.

3.3.13 Summary:

- **On Earth:** own clock runs normally (i.e. according to $d\tau$).
- **Relative to the universal frame:** own clock runs slower due to gravitational and motion influences.

3.3.14 Behavior of a photon in the Schwarzschild metric

A special case occurs when we consider a **photon**. Because a photon always travels at the speed of light *c* and has **no rest mass**, its proper time$d\tau$ along its worldline is equal to zero:

$$d\tau = 0$$

This also follows directly from the Schwarzschild metric:

$$0 = \sigma^2 c^2 dt^2 - \sigma^{-2} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\emptyset^2 \tag{6}$$

From this we can deduce that the distance traveled in space Δdistance is:

$$(\Delta\text{distance})^2 = \sigma^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\emptyset^2 \tag{6b}$$

and the speed of the photon is:

$$c^2 = \left(\frac{\Delta\text{distance}}{\Delta time}\right)^2 = \frac{\sigma^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\emptyset^2}{\sigma^2 dt^2} \tag{6c}$$

3.3.15 Note:

Although a photon would have a "distance of zero" in its own (non-existent) rest frame, an external observer does see a distance traveled along a curved path in space-time.

3.3.16 Special cases:

1. **Radial motion of the photon (only r-direction, $d\tau = d\theta = d\emptyset = 0$)**

Then the previous equation simplifies to:

$$c^2 = \sigma^{-4}\left(\frac{dr}{dt}\right)^2$$

or:

$$\sigma^4 c^2 dt^2 = dr^2 \quad or \quad \frac{\sigma^{-2} dr}{dt} = c \tag{7}$$

2. Circular motion in the equatorial plane ($\theta = \pi/2$)

If the photon is in a circular orbit around the mass *M* (i.e., $d\tau = dr = d\theta = 0$), then:

$$v = c = \frac{r}{\sigma}\frac{d\emptyset}{dt}$$

This shows that the rotational speed $d\emptyset/dt$ depends on the distance *r* and the curvature factor σ .

3.3.17 At a large distance:

If $r = \infty$, then:

$$\sigma \to 1$$

And so:

$$d\tau = dt$$

The time measured along the path of the photon and the coordinate time then coincide—exactly as we would expect in a region without gravity (flat space-time).

3.3.18 In summary:

- For a photon, $d\tau = 0$ always applies.
- The relationship between space and time is completely determined by the curvature factor σ .
- In strongly curved space-time (close to mass), the behavior of a photon deviates significantly from what we intuitively expect in flat space-time.

In general, the motion at infinite distance is rectilinear and uniform, and so the equation becomes:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\emptyset^2 \tag{8}$$

3.3.19 Transformation to Cartesian Coordinates

Schwarzschild's original approach was not in polar **coordinates**, but in **Cartesian coordinates**. Although the Schwarzschild solution is usually presented in spherical coordinates $(r, \theta, \emptyset)$, it can also be expressed in terms of (*x, y, z*).

The transformation of the Schwarzschild metric to Cartesian coordinates results in:

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - (dx^2 + dy^2 + dz^2) - \frac{1-\sigma^2}{\sigma^2 r^2}(xdx + ydy + zdz)^2 \qquad (9)$$

Where:

- $\sigma = \sqrt{1 - \frac{2GM}{c^2 r}}$
- $r = \sqrt{x^2 + y^2 + z^2}$ is the usual radial distance.

3.3.20 Explanation:

The first term $\sigma^2 c^2 dt^2$ describes the time component that depends on gravity. The second term $(dx^2 + dy^2 + dz^2)$ corresponds to flat space-time (Euclidean distance). The third term corrects for the fact that time dilation (and thus curvature) also affects space components, depending on the direction in which we move relative to the mass *M*.

3.3.21 Note on differentiating with respect to t or τ

When using the Schwarzschild metric, we must be alert to the variable with respect to which we differentiate:

- **Time *t*** is the *coordinate time*, as measured by an observer at infinite distance (or in a region without mass).
- **Time τ** is the *proper time*, as measured along the worldline of the moving object.

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \sigma^{-2} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\emptyset^2 \qquad (1a)$$

In formula form (for the plane $\theta = \pi/2$) and dividing by $c^2 d\tau^2$:

$$1 = \sigma^2 \left(\frac{dt}{d\tau}\right)^2 - \sigma^{-2} \left(\frac{dr}{cd\tau}\right)^2 - r^2 \left(\frac{d\emptyset}{cd\tau}\right)^2 \qquad (10)$$

or, by rewriting with partial derivatives with respect to *t*:

$$1 = \sigma^2 \left(\frac{dt}{d\tau}\right)^2 - \sigma^{-2} \left(\frac{dr}{cdt}\frac{dt}{d\tau}\right)^2 - r^2 \left(\frac{d\emptyset}{cdt}\frac{dt}{d\tau}\right)^2$$

Then:

$$1 = \sigma^2 \left(\frac{dt}{d\tau}\right)^2 \left(1 - \frac{1}{\sigma^4}\left(\frac{dr}{cdt}\right)^2 - \frac{r^2}{\sigma^2}\left(\frac{d\emptyset}{cdt}\right)^2\right) \tag{11}$$

From this follows how motion (velocities) and time dilations in curved space-time are related.

3.3.22 Velocity relative to local and universal time

The velocity relative to proper **time** τ is:

$$v_{co}{}^2 = \frac{\sigma^{-2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\emptyset^2}{d\tau^2}$$

and the Schwarzschild metric can be written in terms of v_{co} :

$$c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \frac{\sigma^{-2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\emptyset^2}{d\tau^2} d\tau^2 = \sigma^2 c^2 dt^2 - v_{co}{}^2 d\tau^2$$

Or:

$$c^2 d\tau^2 + v_{co}{}^2 d\tau^2 = \sigma^2 c^2 dt^2$$

Approximation for small velocities ($v_{co} \ll c$) via a Taylor series:

$$d\tau^2 = \frac{\sigma^2}{1 + \left(\frac{v_{co}}{c}\right)^2} dt^2 \approx \sigma^2 \left(1 - \left(\frac{v_{co}}{c}\right)^2\right) dt^2$$

$$d\tau \approx \sigma \sqrt{1 - \left(\frac{v_{co}}{c}\right)^2} \, dt \Longrightarrow d\tau = \frac{\sigma}{\gamma_{co}} dt$$

With $\gamma_{co} = 1/\sqrt{(1 - v_{co}^2/c^2}$.

3.3.23 Summary

- Schwarzschild originally worked in Cartesian coordinates.
- The Schwarzschild metric can be reduced to both spherical and Cartesian form.
- When interpreting motion (e.g., orbits around a mass), it is essential to make a clear distinction between the coordinate time t and the proper timeτ .
- For small velocities, the influence of space-time curvature on the perception of time is small but measurable.

In general, a trajectory takes place in a single plane. In that case, the polar system can always be chosen so that the equatorial plane coincides with the trajectory plane; in that case, $\theta = \pi/2$.

If the trajectory is a circle, so that $\theta = \pi/2$ and r is constant, then $dr = 0$ and the equation becomes:

$$c^2d\tau^2 = \sigma^2c^2dt^2 - r^2d\emptyset^2$$

Additional considerations:

Addendum 1: Interpretation of *ds* as a line segment in space-time

It may be useful to consider *ds* as an infinitely small line segment in space-time, the length of which in meters can be measured by multiplying the travel time of a photon over that segment by the speed of light *c*.

The line segment *ds* is located at the origin of its own moving reference frame. Within that frame, time is the only physical quantity that can be measured directly. The distance is measured via the travel time of the photon.

So:

$$ds = cd\tau$$

where $d\tau$ is the proper time recorded on a moving clock.

Now we introduce a **second reference frame**, for example the Schwarzschild frame, in which there is a central mass *M*. In this frame, we can determine the distance between the line segment and the origin by using external measuring instruments (such as lasers, rods, etc.).

Important:

The time determination in this external frame is indirect: it depends on the relationship given by the Schwarzschild metric and cannot be measured directly with a clock on site.

According to the Schwarzschild metric, the following applies:

$$c^2d\tau^2 = (\mathrm{c}\Delta T)^2 - (\Delta X)^2$$

$$(\mathrm{c}\Delta T)^2 = \left(1 - \frac{2GM}{c^2r}\right)c^2dt^2 = c^2d\tau^2 - (\Delta X)^2$$

Where ΔX is the spatial component, for example:

$$(\Delta X)^2 = \frac{1}{\left(1 - \frac{2GM}{c^2r}\right)}dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\emptyset^2$$

From this it follows:

$$c^2dt^2 = \frac{(\mathrm{c}\Delta T)^2}{\left(1 - \frac{2GM}{c^2r}\right)}$$

The relationship between the theoretical time *dt* and the proper time $d\tau$ can only be determined using this formula.

Addendum 2: Worldline of a Particle in a Co-moving Reference Frame

If we consider a particle in a co-moving (local) reference frame, the particle is at rest relative to that frame. The only path that the particle follows in space-time is along its ownτ axis: its proper time.

However, we can describe the motion of the particle relative to an **external** (possibly moving) reference frame. In that case, we express the position of the particle in coordinates (*t, x, y, z*) of that other frame.

The world line of the particle—the path that the particle follows through space-time—is then entirely a function of τ :

$$\big(t(\tau), x(\tau), y(\tau), z(\tau)\big)$$

The four coordinates are therefore functions of the proper time τ .

3.3.24 Example: Time difference between the poles and the equator

We calculate the time difference on the Earth's surface between the time at the poles and at the equator, caused by relativistic effects.

Based on the Schwarzschild metric:

$$c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \sigma^{-2} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\emptyset^2$$

3.3.25 At the poles

At the poles, the following applies:

- $dr = 0$ (no radial movement),
- $\theta = 0$,
- $d\theta = 0$,
- $\sin\theta = 0$.

Then the Schwarzschild metric becomes:

$$c^2 d{\tau_{poles}}^2 = \sigma^2 c^2 dt^2$$

from which it follows:

$$d\tau_{poles} = \sigma dt$$

3.3.26 At the equator

At the equator, the following applies:

- $dr = 0$,
- $\theta = \pi/2$,
- $d\theta = 0$,
- $\sin\theta = 1$

Then the metric becomes:

$$c^2 d\tau_{equator}{}^2 = \sigma^2 c^2 dt^2 - r^2 d\emptyset^2$$

where $d\emptyset$ describes the rotation around the Earth's axis of rotation.

We rewrite this as:

$$c^2 d\tau_{equator}{}^2 = c^2 d\tau_{poles}{}^2 - r^2 d\emptyset^2$$

Or:

$$c^2 d\tau_{equator}{}^2 = c^2 d\tau_{poles}{}^2 \left\{1 - \frac{r^2}{c^2}\left(\frac{d\emptyset}{d\tau_{poles}}\right)^2\right\}$$

The rotational speed $v_{equator}$ at the equator is:

$$v_{equator} = r\frac{d\emptyset}{d\tau_{poles}}$$

So:

$$c^2 d\tau_{equator}{}^2 = c^2 d\tau_{poles}{}^2 \left\{1 - \frac{v_{equator}{}^2}{c^2}\right\}$$

$$d\tau_{equator} = d\tau_{poles}\sqrt{1 - \frac{v_{equator}{}^2}{c^2}}$$

For small speeds($v \ll c$) , we can approximate this using a Taylor series:

$$d\tau_{equator} \approx d\tau_{poles}\,(1 - \frac{1}{2}\frac{v_{equator}{}^2}{c^2})$$

3.3.27 Practical Calculation

The rotational speed at the equator is approximately:

≈ 1672 km/h or 465 m/s

This results in:

$$\frac{v_{equator}{}^2}{c^2} \approx 2.4 * 10^{-12}$$

Therefore:

$$d\tau_{equator} = d\tau_{poles}\,(1 - 1.2 * 10^{-12})$$

3.3.28 Interpretation

A clock at the equator ticks slightly slower than a clock at the poles. Over a period of 100 years, the difference would amount to approximately:

$$100\ year * 1.2 * 10^{-12} \approx\ 3{,}75 \text{ milliseconds}$$

3.3.29 Conclusion

A person who has lived at the North Pole for 100 years would (theoretically) be 3.75 milliseconds older than someone who has lived at the equator, if all other conditions remain the same.

3.3.30 Addendum 3: The Schwarzschild coefficient and the escape velocity

Let's pay special attention to the factor:

$$1 - \frac{2GM}{c^2 r}$$

This expression is reminiscent of the well-known formula for escape velocity, which determines the minimum speed a mass must have in order to leave another mass (e.g., the Earth).

We will derive this relationship below.

3.3.31 Calculation of the Minimum Escape Velocity

If we consider a mass *m* that is ejected from an object with mass *M* (e.g., the Earth), then:

- The kinetic energy of *m*:

$$E_{kin} = \frac{1}{2} m v^2$$

- The gravitational force experienced by *m*:

$$F = G\frac{Mm}{r^2}$$

where *r* is the distance to the center of mass *M*.

- The work that must be done to bring the mass *m* from *r* to infinity (where the gravitational influence is zero) is:

$$W = \int_r^\infty F ds = \int_r^\infty G\frac{Mm}{s^2} ds = -G\frac{Mm}{s}\Big|_r^\infty = GMm\left(\frac{1}{r} - \frac{1}{\infty}\right) = G\frac{Mm}{r}$$

In order to escape, the kinetic energy must be equal to this work:

$$\frac{1}{2} m v^2 = G\frac{Mm}{r}$$

From which it follows:

$$v^2 = \frac{2GM}{r}$$

$$v = \sqrt{\frac{2GM}{r}}$$

Or conversely:

$$r = \frac{2GM}{v^2}$$

3.3.32 The Maximum Speed: Light

The maximum speed that an object can reach is the speed of light c. If

$$v = c$$

then the corresponding distance becomes:

$$r = \frac{2GM}{c^2}$$

This is known as the **Schwarzschild radius**:

$$r = R_s = \frac{2GM}{c^2}$$

When an object is within this radius, it is impossible to escape—even light cannot escape. In that case, we refer to it as a **black hole**.

3.3.33 Connection with the Schwarzschild metric

In the Schwarzschild metric, this radius appears in the factor:

$$1 - \frac{2GM}{c^2 r}$$

Normally positive, this factor lies between 1 and 0:

Normally, this factor lies between 1 and 0:

- **For large distances** $(r \to \infty)$, the factor approaches 1.
- **Near the Schwarzschild radius**$(r = R_s)$, the factor becomes 0.

When:

$$1 - \frac{2GM}{c^2 r} = 0$$

the object is located exactly at the Schwarzschild radius, or at the **event horizon** of a black hole.

3.3.34 Note

If *r* becomes smaller than R_s , the factor would become negative. What this means physically requires deeper analysis within the relativistic theory of black holes, and goes beyond what we are discussing here.

3.3.35 Key insights

- In general relativity, **geometry replaces force**: mass curves space-time instead of generating a force field.
- The Schwarzschild metric describes this curvature around a **spherically symmetric mass** and applies outside the mass, where $T_{\mu\nu} = 0$.
- A test particle moving in this field **feels no force**, but **follows a** geodesic—the "straightest" path in curved space-time.
- The metric in spherical coordinates:

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \sigma^{-2} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\emptyset^2 \quad \text{with} \quad \sigma = \sqrt{1 - \frac{2GM}{c^2 \mathrm{r}}}$$

- The coordinate time *dt* applies in the asymptotically flat (hypothetical) framework at $r \to \infty$; the proper time $d\tau$ is measured by a clock that moves with the object.

3.3.36 Intuitive explanation

Einstein states: a mass influences the **measurement structure** of space and time itself. A freely falling object does not move "by force," but **follows the path imposed by the geometry of space**-time—similar to a pebble rolling on a curved surface.

The Schwarzschild metric states:

- **Time passes more slowly** closer to a mass (via $\sigma < 1$).
- **Space is stretched** in the radial direction.
- For a moving object, the metric relates $d\tau$ to dt and its velocity v :

$$d\tau = \frac{\sigma}{\gamma} dt \quad \text{with} \quad \gamma = \frac{\mathbf{1}}{\sqrt{(1 - \frac{v^2}{c^2}}}$$

3.3.37 Table: Concepts and physical quantities from the Schwarzschild metric

Quantity	Physical interpretation
$\sigma = \sqrt{1 - \frac{2GM}{c^2 \mathrm{r}}}$	Gravitational time dilation factor
$d\tau$	Proper time: measured by a local clock
dt	Coordinate time: measured in asymptotic plane reference frame
v	Local velocity, derived from spatial coordinates and dt
$ds^2 = c^2 d\tau^2$	Four-dimensional interval (invariant): time and space element
$R_s = \frac{2GM}{c^2}$	Schwarzschild radius: where $\sigma = 0$, event horizon

3.3.38 Specific cases and effects

- **Photons**:
    - Following a trajectory with $d\tau = 0$: they do not experience their own time.
    - The **effective speed of light in coordinate time** is less than *c* (but locally, *v* remains= *c*).
- **Object at rest with mass**:
    - Time dilation via $d\tau = \frac{\sigma}{\gamma} dt$: the smaller *r*, the slower the clock.
- **Object in motion (e.g., circular orbit)**:
    - Both gravitational and kinematic time dilation play a role.
- **Equation for velocity** in Schwarzschild coordinates:

$$v^2 = \sigma^{-4} \left(\frac{dr}{dt}\right)^2 + \sigma^{-2} r^2 \left(\frac{d\theta}{dt}\right)^2 + \sigma^{-2} r^2 \sin^2\theta \left(\frac{d\emptyset}{dt}\right)^2 \tag{3a}$$

3.3.39 Application: Time on Earth vs. Time at infinite distance

- Time on Earth slows down relative to universal coordinate time, due to both:
    - gravity (via σ),
    - Earth's rotation (viaγ).
- Result: in 100 years, a clock at the poles≈ will be 3.75 ms ahead of a clock at the equator (approximately).

3.3.40 Black holes and escape velocity

- From Newton's mechanics it follows that:

$$v_{escape} = \sqrt{\frac{2GM}{\mathrm{r}}} \quad \Longrightarrow \quad R_s = \frac{2GM}{c^2}$$

- At $r = R_s$, $\sigma \rightarrow$ **event horizon**: nothing can escape.
- The Schwarzschild metric explicitly contains this boundary.

3.3.41 Final insight

The Schwarzschild metric provides **directly measurable predictions**:

- Time dilation (including GPS corrections),
- Light deflection (as measured during solar eclipses),
- Mercury's perihelion precession,
- Conditions for the formation of black holes.

The Schwarzschild metric is therefore not an abstract mathematical object, but a physical machine that tells us **how clocks tick, how light bends, and how masses move—purely on the basis of the geometry of space and time.**

3.4 Experiments: Confirmation of General Relativity

The general theory of relativity is not only an elegant mathematical theory, but is also strongly supported by experiments and observations. Many of these experiments use the **Schwarzschild-solution** as the basis for their theoretical predictions.

The following experiments are discussed in this work:

1. **Hafele-Keating experiment (1971)** (see chapter4.1)

- **Description:** Atomic clocks were flown around the Earth in airplanes, both eastward and westward, and compared with clocks on the ground.
- **Result:** The measured time differences corresponded exactly to the predictions of general relativity (gravitational time dilation and motion time dilation).
- **Relationship to Schwarzschild metric:** The time dilation due to gravity is derived directly from the Schwarzschild solution.

2. **Motion of particles in a gravitational field** (see section4.2)

- **Description:** The orbits of satellites, planets, and other objects are tracked accurately.

- **Result:** The observed orbits correspond to the predictions of Schwarzschild geometry, including small deviations from the classical (Newtonian) predictions.

3. **Deflection of light near masses** (see section4.3)

- **Description:** During solar eclipses, measurements were taken of how the light from stars is deflected by the gravity of the sun.
- **Result:** The measured deflection (by Eddington in 1919 and many later experiments) corresponds exactly to the value predicted by Schwarzschild metric.
- **Physical significance:** Proves that light itself is affected by the curvature of space-time.

4. **Precession of the perihelia (Mercury)** (see section4.4)

- **Description:** Mercury's orbit slowly rotates around the sun; this precession cannot be fully explained by Newtonian gravity.
- **Result:** The remaining precession is explained exactly by the Schwarzschild solution.
- **Historical significance:** This was one of the first major successes of general relativity.

5. **Shapiro time delay** (see section4.5)

- **Description:** Radio waves traveling past the Sun take longer than expected in flat space-time.
- **Result:** The extra time delay corresponds to the prediction of the Schwarzschild metric.
- **Application:** Used in radar and communications satellites.

6. **Trajectory of a bullet in a strong gravitational field** (see section4.8)

- **Description:** Simulations and measurements of objects moving at high speed near massive objects.
- **Result:** The trajectories deviate from Newtonian predictions but correspond to the Schwarzschild predictions.

Conclusion

In all these experiments, the results are in excellent agreement with the predictions of general relativity, as derived from the Schwarzschild metric. This provides powerful confirmation of the correctness of Einstein's theory.

Key insight

General relativity is not only a mathematically elegant theory, but has also been experimentally confirmed with high precision. The Schwarzschild metric is the key to understanding most classical tests of gravity.

Part IV – Experiments and Verifications

4 Experiments Confirming Einstein's Theory

In this chapter, we discuss a series of experiments that empirically support Einstein's general theory of relativity. A central tool in the analysis of these experiments is the Schwarzschild solution of Einstein's field equations.

The following experiments are discussed:

- The Hafele–Keating experiment (see Chapter4.1)
- The motion of particles in a gravitational field (see chapter4.2)
- The deflection of light in the vicinity of masses (see section4.3)
- The precession of the perihelia of planets, particularly Mercury (see section4.4)
- The Shapiro time delay (see section4.5)
- The calculation of the trajectory of a bullet in a strong gravitational field (see section4.8)

Together, these experiments provide powerful evidence for the validity of general relativity. In each case, the Schwarzschild metric provides a mathematical framework that can be used to accurately explain the observed phenomena.

4.1 Experiment 1 - The Hafele & Keating Experiment with the Schwarzschild Equation

Derivation based on: *A Hafele & Keating-like thought experiment*, by Paul B. Andersen, October 16, 2008 (Andersen, 2008).

The famous experiment by Hafele and Keating tested quantitative predictions of the theory of relativity, specifically time dilation resulting from both motion (special theory of relativity) and gravity (general theory of relativity).

In this experiment, two planes were equipped with cesium clocks and flown simultaneously in opposite directions around the Earth. In addition, a third cesium clock remained at a fixed location on Earth (in Washington). The results showed that the clocks on board underwent different time dilation effects, depending on their direction of movement and position relative to the Earth.

The clock in the eastbound aircraft moved with the Earth's rotation. As a result, this clock had a higher speed relative to the non-rotating center of the Earth than the ground clock. This led to a stronger time dilation: the clock ran slow. In contrast, the westbound plane moved against the Earth's rotation, resulting in a lower speed relative to the Earth's center, and thus a weaker time dilation: this clock ran fast. This difference in the passage of time shows that the progress of time depends on the movement of the observer—an effect that was predicted as early as 1905 by Einstein in his original article on the special theory of relativity.

All three clocks are moving eastward. Even though the airplane flying westward is moving westward relative to the air, the air is moving eastward due to the Earth's rotation.
Source: (Crowell, March 11, 2018)

Purpose and Design

- **Objective:** Direct experimental test of Einstein's predictions for time dilation due to both motion (special relativity) and gravity (general relativity).
- **Design:** Cesium clocks were flown in airplanes eastward and westward around the Earth, while a reference clock remained on Earth. The time differences between these clocks were measured and compared with the theoretical predictions.

Theoretical Framework: Schwarzschild Metric

The **Schwarzschild metric** describes the space-time outside a spherically symmetric massive body such as the Earth:

$$c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\emptyset^2 \tag{1}$$

The following applies:

- t: the coordinate time, measured by a hypothetical clock outside any gravitational field;
- τ: the proper time, measured by a moving clock at position *r*;
- *r*: distance to the center of the Earth;
- θ: angle of latitude relative to the North Pole;
- $\emptyset$: angle of longitude relative to a fixed meridian;
- *G*: the gravitational constant;

- *M*: the mass of the Earth;
- *c*: the speed of light.

The Schwarzschild metric uses a universal (spherical) coordinate system with its origin at the center of mass of the Earth. The Earth rotates within this coordinate system. Changes in the angles θ and $\emptyset$ describe motion across the Earth's surface. Small changes in time and space are denoted by *dt*, *dr*, $d\theta$, and $d\emptyset$, respectively.

Note that *dt* is the time change for a hypothetical observer far away from gravitational influences; it is not a directly measured time but a theoretical coordinate time. The actual time measured by a clock at a given location is $d\tau$, the proper time.

We will use the Schwarzschild metric to derive an approximate formula that describes the time dilation of the clocks, based on their position and motion. We will then also give the complete (exact) solution. Although the latter is more complex, it is easily manageable with the help of computer programs such as Excel and provides an accurate result.

4.1.1 Approximate Formula for Time Dilation

We approximate the situation in which the clocks are in circular orbits around the Earth: either at sea level or at a certain height above the Earth's surface. Because the orbits are circular, $dr = 0$. Furthermore, we assume that the motion takes place in the plane of the equator, so that $\theta = \pi/2$ is constant and therefore $d\theta = 0$.

This simplifies equation (1) to:

$$c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - r^2 d\emptyset^2 \tag{2}$$

$$d\tau^2 = \left(\left(1 - \frac{2GM}{c^2 r}\right) - \frac{r^2}{c^2}\left(\frac{d\emptyset}{dt}\right)^2\right) dt^2 \tag{3}$$

If we use $v = r\frac{d\emptyset}{dt}$, we get:

$$d\tau = \sqrt{\left(1 - \frac{2GM}{c^2 r} - \frac{v^2}{c^2}\right)} dt \tag{4}$$

Because the terms $\frac{2GM}{c^2 r}$ and $\frac{v^2}{c^2}$ are very small compared to 1, we can apply a first-order approximation with a Taylor series:

$$d\tau \approx \left(1 - \frac{GM}{c^2 r} - \frac{v^2}{2c^2}\right) dt \tag{5}$$

Since *r* and v are constant, we can easily integrate:

$$\tau = \left(1 - \frac{GM}{c^2 r} - \frac{v^2}{2c^2}\right) t + \tau(0) \tag{6}$$

It is interesting to compare the local time τ of different clocks. We take the clock on the Earth's surface as a reference. This clock has a speed of v_1 due to the Earth's rotation and is located at a distance of r_1 from the center of the Earth. For this clock, the following applies:

$$d\tau_1 = \left(1 - \frac{GM}{c^2 r_1} - \frac{{v_1}^2}{2c^2}\right) dt \tag{7}$$

For a clock in an airplane (with speed v_2 and distance r_2):

$$d\tau_2 = \left(1 - \frac{GM}{c^2 r_2} - \frac{{v_2}^2}{2c^2}\right) dt \tag{8}$$

We now want to express the passage of time of clock 2 relative to clock 1. To do this, we use an approximation for the ratio of the time units:

$$d\tau_2 = \frac{\left(1 - \frac{GM}{c^2 r_2} - \frac{{v_2}^2}{2c^2}\right)}{\left(1 - \frac{GM}{c^2 r_1} - \frac{{v_1}^2}{2c^2}\right)} d\tau_1$$

Using $(1-\varepsilon)^{-1} \approx 1 + \varepsilon$, we get:

$$d\tau_2 \approx \left(1 - \frac{GM}{c^2 r_2} - \frac{{v_2}^2}{2c^2}\right)\left(1 + \frac{GM}{c^2 r_1} + \frac{{v_1}^2}{2c^2}\right) d\tau_1$$

$$d\tau_2 \approx \left(1 + \frac{GM}{c^2 r_1} + \frac{{v_1}^2}{2c^2} - \frac{GM}{c^2 r_2}\left(1 + \frac{GM}{c^2 r_1} + \frac{{v_1}^2}{2c^2}\right) - \frac{{v_2}^2}{2c^2}\left(1 + \frac{GM}{c^2 r_1} + \frac{{v_1}^2}{2c^2}\right)\right) d\tau_1$$

Since the terms $\frac{GM}{c^2 r_1}$, $\frac{{v_1}^2}{2c^2}$, $\frac{GM}{c^2 r_1}$ and $\frac{{v_1}^2}{2c^2}$ are very small, their products can be neglected, from which it follows:

$$d\tau_2 \approx \left(1 + \frac{GM}{c^2 r_1} + \frac{{v_1}^2}{2c^2} - \frac{GM}{c^2 r_2} - \frac{{v_2}^2}{2c^2}\right) d\tau_1$$

$$d\tau_2 \approx \left(1 + \frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{{v_1}^2 - {v_2}^2}{2c^2}\right) d\tau_1 \tag{9}$$

If we assume that both clocks start at $\tau_1 = \tau_2 = 0$, then the integration is immediate:

$$\tau_2 \approx \left(1 + \frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{{v_1}^2 - {v_2}^2}{2c^2}\right) \tau_1 \tag{10}$$

Comparison between Clocks

For a clock on the Earth's surface (r_1, v_1) and a clock in an airplane (r_2, v_2) :

$$\tau_2 - \tau_1 \approx \left(\frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{{v_1}^2 - {v_2}^2}{2c^2}\right) \tau_1 \tag{11}$$

Suppose that clock 1 is located on the Earth's surface with radius *R*, and clock 2 is in an airplane at altitude *h*, then r_2= *R+h*. Because $h \ll R$, we can approximate:

$$\frac{1}{R} - \frac{1}{R+h} \approx \frac{\mathrm{h}}{\mathrm{R}^2} \Rightarrow \frac{GM}{c^2}\left(\frac{1}{R} - \frac{1}{R+h}\right) \approx \boldsymbol{\frac{gh}{c^2}}$$

So:

$$\tau_2 - \tau_1 \approx \left(\frac{GM}{c^2}\left(\frac{1}{R} - \frac{1}{R+h}\right) + \frac{{v_1}^2 - {v_2}^2}{2c^2}\right)\tau_1 \approx \left(\frac{GM}{c^2}\left(\frac{\mathrm{h}}{\mathrm{R}^2}\right) + \frac{{v_1}^2 - {v_2}^2}{2c^2}\right)\tau_1 \qquad (12)$$

If we assume that $\frac{h}{R} \ll 1$ and the gravitational acceleration is $g = \frac{GM}{R^2}$, then:

$$\boldsymbol{\tau_2 - \tau_1 = \left(\frac{gh}{c^2} - \frac{{v_2}^2 - {v_1}^2}{2c^2}\right)\tau_1} \qquad (13)$$

Since $v_1 = v_{eart\,h}$ (rotational speed of the Earth) and $v_2 = v_{plane} + v_{eart\,h}$ (speed of the airplane relative to the coordinate system), we get:

$${v_1}^2 - {v_2}^2 = {v_{eart\,h}}^2 - \left(v_{plane} + v_{eart\,h}\right)^2 = {v_{eart\,h}}^2 - {v_{plane}}^2 - 2v_{eart\,h}v_{plane} - {v_{eart\,h}}^2$$

$${v_1}^2 - {v_2}^2 = -{v_{plane}}^2 - 2v_{eart\,h}v_{plane} = -v_{plane}\left(v_{plane} + 2v_{eart\,h}\right)$$

Substituting into (13) gives:

$$\boldsymbol{\tau_{plane} - \tau_{earth} = \left(\frac{gh}{c^2} - \frac{v_{plane}\left(v_{plane} + 2v_{earth}\right)}{2c^2}\right)\tau_{earth}} \qquad (14)$$

This equation is therefore derived entirely from the Schwarzschild equation with a few approximations.
This formula corresponds to the approximation used in the original Hafele–Keating experiment and is therefore consistently derived from the Schwarzschild equation, including the necessary approximations.

Note 1:

If the speed v_2 of the aircraft is a ground speed, then, as an approximation at altitude *h,* we can say:

$$v_2 = \frac{R+h}{R}\left(v_{plane} + v_{eart\,h}\right)$$

In that case, formula (13) would be adjusted accordingly.

Note 2:

A more accurate approximation is discussed in the next chapter, where the velocities v_1 and v_2 are derived more specifically based on the coordinate system used.

4.1.2 Elaboration of V_1 and V_2 in Equation (13)

The velocity v_1 in equation 3 1 1 13 is the velocity of a stationary point on the equator of the Earth's surface. This is expressed as:

$$v_1 = r_1\frac{d\emptyset}{dt}$$

where dt is the coordinate time in the 'universal' reference frame. However, because measurements on the Earth's surface are made with respect to the local proper time τ , a conversion is necessary. Due to the chain rule, the following applies:

$$v_{1t} = r_1\frac{d\emptyset}{dt} = r_1\frac{d\emptyset}{d\tau}\frac{d\tau}{dt} = v_{1\tau}\frac{d\tau}{dt} \qquad (14a)$$

We use the Schwarzschild metric (in the equatorial plane and with *dr = 0*) to find $d\tau/dt$:

$$c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r_1}\right) c^2 dt^2 - {r_1}^2 \left(\frac{d\emptyset}{d\tau}\right)^2 d\tau^2$$

Reordering yields:

$$\left(1 + \frac{{r_1}^2}{c^2}\left(\frac{d\emptyset}{d\tau}\right)^2\right) d\tau^2 = \left(1 - \frac{2GM}{c^2 r_1}\right) dt^2$$

We now define:

$$\sigma^2 = 1 - \frac{2GM}{c^2 r}$$

Then the following applies:

$$\left(1 + \frac{{v_{1\tau}}^2}{c^2}\right) d\tau^2 = \left(1 - \frac{2GM}{c^2 r_1}\right) dt^2 = {\sigma_1}^2 dt^2$$

$$\left(\frac{d\tau}{dt}\right)^2 = \frac{{\sigma_1}^2}{\left(1 + \frac{{v_{1\tau}}^2}{c^2}\right)} \qquad (14b)$$

Substitution in (14a) gives:

$${v_{1t}}^2 = {v_{1\tau}}^2 \left(\frac{d\tau}{dt}\right)^2 = {v_{1\tau}}^2 \frac{{\sigma_1}^2}{\left(1 + \frac{{v_{1\tau}}^2}{c^2}\right)}$$

The expression shows that $d\tau/dt$ —and thus the conversion between coordinate time and proper time—depends on $v_{1\tau}$, the rotational speed of the Earth measured in proper time on the Earth's surface.

Now consider an airplane flying eastward. The total speed at ground level is then:

$$v_{1\tau\_plane} = v_{plane\ \_\tau} + v_{1\tau\_eart\ h} = r_1 \frac{d\emptyset}{d\tau}$$

where:

- $v_{plane\ \_\tau}$ is the speed of the airplane relative to the Earth's surface, measured in local time
- $v_{1\tau\_eart\ h}$ is the rotational speed of a stationary point on Earth, also measured in local time

The corresponding angular velocity in the universal frame is then:

$$r_1 \frac{d\emptyset}{dt} = r_1 \frac{d\emptyset}{d\tau}\frac{d\tau}{dt} = \left(v_{plane\ \_\tau} + v_{1\tau\_eart\ h}\right)\frac{d\tau}{dt} = \left(v_{plane\ \_\tau} + v_{1\tau\_eart\ h}\right)\frac{\sigma_1}{\sqrt{1 + \frac{{v_{1\tau\_eart\ h}}^2}{c^2}}}$$

$$\frac{d\emptyset}{dt} = \left(v_{plane\ \_\tau} + v_{1\tau\_eart\ h}\right)\frac{\sigma_1}{r_1\sqrt{1 + \frac{{v_{1\tau\_eart\ h}}^2}{c^2}}}$$

Here we have calculated the rotational speed (angular velocity) in the universal frame. This is valid for every level, or every distance from the center. But the speed itself is determined by *r* times this rotational speed.

Speed at the level of the aircraft

$v_{plane\ \_\tau}$ is the measured speed of the aircraft at ground level and relative to its own time, which is the only time available at that level. $v_{eart\,h\_\tau}$ is the (rotational) speed of a stationary point on Earth relative to the universal frame, but measured with its own time at ground level.

The speed of the aircraft in the universal frame at altitude r_2 is:

$$v_{2t} = r_2 \frac{d\emptyset}{dt} = \frac{r_2}{r_1} \frac{\sigma_1 \left(v_{plane\ \_\tau} + v_{1\tau\_eart\,h}\right)}{\sqrt{1 + \frac{{v_{1\tau\_eart\,h}}^2}{c^2}}}$$

We split this speed into an 'earth rotation' and an 'airplane' component:

$$v_{2t} = v_{2t\_eart\,h} + v_{2t\_plane}$$

With:

$$v_{2t\_eart\,h} = \frac{r_2}{r_1} \frac{\sigma_1 v_{1\tau\_eart\,h}}{\sqrt{1 + \frac{{v_{1\tau\_eart\,h}}^2}{c^2}}}$$

And:

$$v_{2t\_plane} = v_{2t} - v_{2t\_eart\,h} = \frac{r_2}{r_1} \frac{\sigma_1 v_{plane\ \_\tau}}{\sqrt{1 + \frac{{v_{1\tau\_eart\,h}}^2}{c^2}}}$$

Summary of the result:

The conversion of speed at ground level to the universal frame:

$$v_{1t\_eart\,h} = v_{1\tau\_eart\,h} \frac{\sigma_{eart\,h}}{\sqrt{1 + \frac{{v_{1\tau\_eart\,h}}^2}{c^2}}} \tag{15}$$

The speed v_2 of the aircraft in the universal frame is:

$$v_{2t} = \frac{r_2}{r_1} \frac{\sigma_{eart\,h}\left(v_{plan\,e_\tau} + v_{eart\,h_\tau}\right)}{\sqrt{1 + \frac{{v_{1\tau_{eart\,h}}}^2}{c^2}}} \tag{16}$$

Substitution in equation (3.1.1. 13):

$$\tau_2 - \tau_1 = \left(\frac{gh}{c^2} - \frac{{v_2}^2 - {v_1}^2}{2c^2}\right) \tau_1 \tag{13}$$

becomes:

$$\tau_2 - \tau_1 = \left(\frac{gh}{c^2} - \frac{{\sigma_{earth}}^2}{1 + \frac{{v_{1\tau_{earth}}}^2}{c^2}} \cdot \frac{1}{2c^2} \left[\left(\frac{R+h}{R} \right)^2 \left(v_{plane_\tau} + v_{earth_\tau} \right)^2 - {v_{1\tau\_earth}}^2 \right] \right) \tau_1 \tag{17}$$

This equation describes the time dilation between a clock on the Earth's surface and a clock on board an airplane, taking into account both gravitational and velocity-dependent effects, all based on locally measurable quantities.

4.1.3 The Exact Derivation

Instead of an approximation, we now make an exact derivation, based entirely on the Schwarzschild metric.

We start with equation (4)

$$d\tau = \sqrt{\left(1 - \frac{2GM}{c^2 r} - \frac{v^2}{c^2}\right)} dt \tag{4}$$

Since *r* and v are constant, the integration is simple:

$$\tau = \sqrt{\left(1 - \frac{2GM}{c^2 r} - \frac{v^2}{c^2}\right)} t + \tau(0) \tag{6a}$$

The goal is to compare the proper time of different clocks with each other. As a reference, we take the clock on the Earth's surface. Other clocks are located in airplanes, at higher altitudes and at different speeds. Even the clock on Earth has a non-zero speed due to the Earth's rotation.

For the clock on the Earth's surface (radius r_1 , speed v_1), the following applies:

$$d\tau_1 = \sqrt{\left(1 - \frac{2GM}{c^2 r_1} - \frac{{v_1}^2}{c^2}\right)} dt \tag{7a}$$

For the clock in the airplane (radius r_2 , speed v_2):

$$d\tau_2 = \sqrt{\left(1 - \frac{2GM}{c^2 r_2} - \frac{{v_2}^2}{c^2}\right)} dt \tag{8a}$$

To find the ratio between the two proper times, we divide these expressions:

$$d\tau_2 = \sqrt{\frac{\left(1 - \frac{2GM}{c^2 r_2} - \frac{{v_2}^2}{c^2}\right)}{\left(1 - \frac{2GM}{c^2 r_1} - \frac{{v_1}^2}{c^2}\right)}} d\tau_1 \tag{9a}$$

With equal start times $(\tau_2(0) = \tau_1(0) = 0)$, the solution is:

$$\tau_2 = \sqrt{\frac{\left(1 - \frac{2GM}{c^2 r_2} - \frac{{v_2}^2}{c^2}\right)}{\left(1 - \frac{2GM}{c^2 r_1} - \frac{{v_1}^2}{c^2}\right)}}\,\tau_1 \qquad (10a)$$

Time difference between two clocks

The time difference between the two clocks is:

$$\tau_2 - \tau_1 = \left(\sqrt{\frac{\left(1 - \frac{2GM}{c^2 r_2} - \frac{{v_2}^2}{c^2}\right)}{\left(1 - \frac{2GM}{c^2 r_1} - \frac{{v_1}^2}{c^2}\right)}} - 1\right)\tau_1 \qquad (11a)$$

Suppose $r_1 = R$, the radius of the Earth, and $r_2 = R + h$, the altitude of the aircraft, then this becomes:

$$\tau_2 - \tau_1 = \left(\sqrt{\frac{\left(1 - \frac{2GM}{c^2 (R+h)} - \frac{{v_2}^2}{c^2}\right)}{\left(1 - \frac{2GM}{c^2 R} - \frac{{v_1}^2}{c^2}\right)}} - 1\right)\tau_1 \qquad (12a)$$

Speeds and gravitational effects

In this expression:

- v_1 is the rotational speed of a point on the Earth's surface (towards the east),
- v_2 is the speed of the airplane relative to the universal frame of reference.

Both speeds were previously derived in equation 14b and equation 15b of section4.1.5

The time difference between the aircraft and the Earth is therefore:

$$\tau_{plane} - \tau_{eart\,h} = \left(\sqrt{\frac{\left(1 - \frac{2GM}{c^2 (R+h)} - \frac{{v_2}^2}{c^2}\right)}{\left(1 - \frac{2GM}{c^2 R} - \frac{{v_1}^2}{c^2}\right)}} - 1\right)\tau_{eart\,h} \qquad (14b)$$

Using the Schwarzschild radius $R_s = \frac{2GM}{c^2}$, we rewrite this as:

$$\tau_{plane} - \tau_{eart\,h} = \left(\sqrt{\frac{\left(1 - \frac{R_s}{(R+h)} - \frac{{v_2}^2}{c^2}\right)}{\left(1 - \frac{R_s}{R} - \frac{{v_1}^2}{c^2}\right)}} - 1\right)\tau_{eart\,h} \qquad (15b)$$

Conclusion

This equation is an exact expression, derived directly from the Schwarzschild metric. It shows how the difference in proper time between a clock on Earth and a clock in an airplane is influenced by:

- **Gravitational time dilation:** Clocks at higher altitudes (weaker gravity) run faster.
- **Kinematic time dilation:** Clocks that move faster run slower.

Calculations based on the experiments performed:

	PaulAnderson	Re_Spec_92	H&K
Vplane_ground_east_tau	232.55	670	173.98
Vplane_ground_West_tau	-232.55	-670	-124.43
Vplane2_east in dt	232.88	672.00	174.19
Vplane2_west in dt	-232.88	-672.00	-124.62
V_earth_tau	464.58	464.58	464.58
V_earth_t	464.58	464.58	464.58
V_earth_east on plane level dt	465.24	465.97	465.14
V_earth_west on plane level dt	465.24	465.97	465.28
H_east	9000	19000	7664
H_west	9000	19000	9526
t_earth	172328	59746.528	172328
Result (formula 7.1.13):			
Grav_delay (ns)_East	169.46	124.03	144.31
Kin_delay (ns)_East	-260.32	-358.69	-184.94
Total_East	-9.09E-08	-2.35E-07	-4.06E-08
Grav_delay (ns)_West	169.46	124.03	179.37
Kin_delay (ns)_West	155.16	57.63	95.67
Total_West	3.25E-07	1.82E-07	2.75E-07
Exact (Formula: 7.3.15):			
Total_East (ns)	-9.11E-08	-2.35E-07	-4.08E-08
Total_West	3.24E-07	1.81E-07	2.75E-07
diff east	2.35E-10	3.63E-10	1.56E-10
diff west	2.18E-10	3.67E-10	2.58E-10
diff east in %	-0.26	-0.15	-0.38
diff west in %	0.07	0.20	0.09
sidereal day: 23.9344696hr	86164.1	86164.1	86164.1
Lightvelocity	299792458	299792458	299792458
G	6.67E-11	6.67E-11	6.67E-11
M_earth	5.97E+24	5.97E+24	5.97E+24
R_earth	6371000	6371000	6371000
Schwarzschild radius Rs:	8.87E-03	8.87E-03	8.87E-03

Conclusion:

The approximations are correct within an accuracy of 0.4%.

Practical Application

- **Earth's rotational speed (equator)**:v_{earth} is approximately $464.58\ \mathrm{m/s}$ (based on the sidereal day).
- **Aircraft:** Speed relative to the Earth's surface, corrected for altitude.
- **Altitude effect:** *h* is typically several kilometers, *R* (Earth) approximately $6.371 * 10^6\mathrm{m}$.

Results and Interpretation

- **Eastward-flying clock:** Greater speed relative to the Earth's center→ stronger kinematic time dilation→ clock runs slow.
- **Clock flying westward:** Lower speed relative to the Earth's center→ weaker time dilation→ clock runs fast.
- **Gravitational effect:** Clocks at higher altitudes (airplanes) run faster due to weaker gravity.

Experimental outcome

- The measured time differences corresponded exactly to the predictions of general relativity, with an accuracy of less than 0.4%.
- Both the approximate and exact formulas (derived from the Schwarzschild metric) are consistent with the observations.

Summary

- The Hafele–Keating experiment is a direct, quantitative confirmation of Einstein's theory of relativity.
- The Schwarzschild metric provides the mathematical framework for explaining these time dilation effects.
- **Both effects—motion and gravity—are essential and are measured and explained simultaneously.**

4.1.4 The Speed of a Stationary Point on the Equator at the Earth's Surface

To calculate the speed of a stationary point on the equator, we must first determine the rotation time of the Earth: the *sidereal day*.

Sidereal day versus 24-hour day

A normal day or 24-hour period is the time between two consecutive highest positions of the sun in the sky. This time is based on the **solar cycle**, not on the actual rotation of the Earth.

Due to the Earth's annual orbit around the sun, the Earth makes **one extra rotation** relative to the fixed starry sky in a year. In one year (365.25 solar days), the Earth therefore rotates **366.25 times** around its axis relative to the stars.

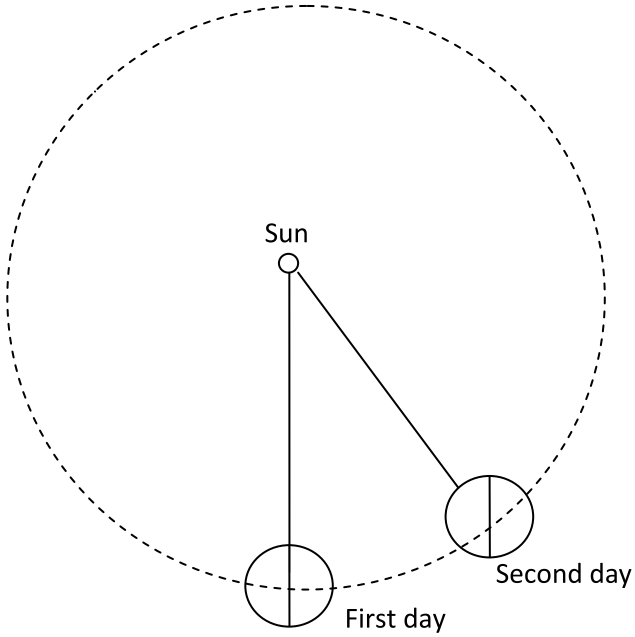
This gives us the duration of one sidereal day:

$$T_{sidereal} = \frac{365.25}{366.25} * 24 * 3600 = 86164.1 \; seconds.$$

This is converted to:

$$\frac{86164.1}{3600} = 23.93447 \text{ hour}$$

⇨ Or 23 hours, 56 minutes, and 4 seconds.



Speed at the equator

With the radius of the Earth R=6371 km, or $6.371 * 10^6$ m, we can calculate the circumference at the equator:

$$Circumference = 2\pi R = 2\pi \times 6{,}371 \times 10^6 \approx 4{,}003 \times 10^7 \ m$$

The speed of a stationary point on the equator (relative to a non-rotating frame of reference) is then:

$$v_{earth} = \frac{2\pi R}{T_{sidereal}} = \frac{4.003 * 10^7}{86164.1} = 464.58 \ m/s$$

For comparison: If we incorrectly assume a 24-hour day, we get:

$$v = \frac{2\pi R}{86400} = 463.3 \ m/s$$

Summary

- A **sidereal day** lasts approximately **23 hours, 56 minutes, and 4 seconds**.
- The speed of a stationary point at the **equator** is approximately **464.58 m/s**.
- The difference between a sidereal day and a solar day leads to a measurable difference in speed, which is important for relativistic calculations such as in the Hafele–Keating experiment.

4.1.5 Correction to derivation based on Paul Anderson

In Anderson's original derivation, the speed of the aircraft is entered relative to the Earth's surface. However, in his formula 3.1.1.3, this speed is expressed relative to the coordination time *dt,* while the clocks in motion measure proper time $d\tau$. This requires a correction: the speed of the object must be expressed relative to its own clock, i.e., via $d\tau$.

Starting point: The complete Schwarzschild relation

We take equation 2 from section4.1.1as our starting point, without approximation:

$$c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - r^2 d\emptyset^2 \tag{2}$$

Dividing by c^2 gives:

$$d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \frac{r^2}{c^2}\left(\frac{d\emptyset}{d\tau}\right)^2 d\tau^2 \tag{3b}$$

$$d\tau^2 \left[1 + \frac{r^2}{c^2}\left(\frac{d\emptyset}{d\tau}\right)^2\right] = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 \tag{3c}$$

$$d\tau = \sqrt{\frac{\left(1 - \frac{2GM}{c^2 r}\right)}{1 + \frac{r^2}{c^2}\left(\frac{d\emptyset}{d\tau}\right)^2}}\, dt \tag{4b}$$

With:

$$v_\tau = r\frac{d\emptyset}{d\tau} \tag{4c}$$

So:

$$d\tau_1 = \sqrt{\frac{\left(1 - \frac{2GM}{c^2 r_1}\right)}{1 + \frac{{v_1}^2}{c^2}}}\, dt \tag{7b}$$

$$d\tau_2 = \sqrt{\frac{\left(1 - \frac{2GM}{c^2 r_2}\right)}{1 + \frac{{v_2}^2}{c^2}}}\, dt \tag{8b}$$

$$d\tau_2 = \sqrt{\frac{\left(1 - \frac{2GM}{c^2 r_2}\right)\left(1 + \frac{{v_1}^2}{c^2}\right)}{\left(1 - \frac{2GM}{c^2 r_1}\right)\left(1 + \frac{{v_2}^2}{c^2}\right)}}\, d\tau_1 \tag{9b}$$

This results in the equation between both proper times:

$$\tau_2 = \sqrt{\frac{\left(1-\frac{2GM}{c^2 r_2}\right)\left(1+\frac{{v_1}^2}{c^2}\right)}{\left(1-\frac{2GM}{c^2 r_1}\right)\left(1+\frac{{v_2}^2}{c^2}\right)}}.\tau_1 \qquad (10b)$$

If we set $\tau_1 = \tau_{earth}$ (clock at sea level)and $\tau_2 = \tau_{plane}$, we get:

$$\boldsymbol{\tau_{plane}} - \boldsymbol{\tau_{earth}} = \left(\sqrt{\frac{\left(1-\frac{2GM}{c^2 r_2}\right)\left(1+\frac{{v_1}^2}{c^2}\right)}{\left(1-\frac{2GM}{c^2 r_1}\right)\left(1+\frac{{v_2}^2}{c^2}\right)}} - 1\right)\boldsymbol{\tau_{earth}} \qquad (11b)$$

Here, $r_1 = R$, is the radius of the Earth. The distance from the clock in an airplane is then $R + h$:

$$\boldsymbol{\tau_{plane}} - \boldsymbol{\tau_{earth}} = \left(\sqrt{\frac{\left(1-\frac{2GM}{c^2 (R+h)}\right)\left(1+\frac{{v_{earth}}^2}{c^2}\right)}{\left(1-\frac{2GM}{c^2 R}\right)\left(1+\frac{{v_2}^2}{c^2}\right)}} - 1\right)\boldsymbol{\tau_{earth}} \qquad (14b)$$

Or with Schwarzschild radius $R_s = \frac{2GM}{c^2}$:

$$\boldsymbol{\tau_{plane}} - \boldsymbol{\tau_{earth}} = \left(\sqrt{\frac{\left(1-\frac{R_s}{(R+h)}\right)\left(1+\frac{{v_{earth}}^2}{c^2}\right)}{\left(1-\frac{R_s}{R}\right)\left(1+\frac{{v_2}^2}{c^2}\right)}} - 1\right)\boldsymbol{\tau_{earth}} \qquad (15b)$$

Relative speed at cruising altitude

The ground speed of the airplane (relative to the Earth's surface) must be converted to a coordinate-independent speed at flight altitude:

$$v_2 = \left(v_{earth} + v_{plane\ \ relative\ \ to\ earth\ point}\right) \cdot \frac{R+h}{R}$$

So far, the formula has been without approximations.

After first-order Taylor approximations of equation (14b), as done previously, the result is:

$$\boldsymbol{\tau_{plane}} - \boldsymbol{\tau_{earth}} = \left(\left(1-\frac{GM}{c^2 (R+h)}\right)\left(1+\frac{GM}{c^2 R}\right)\left(1+\frac{{v_{earth}}^2}{2c^2}\right)\left(1-\frac{{v_2}^2}{2c^2}\right) - 1\right)\boldsymbol{\tau_{earth}} \qquad (16)$$

$$\boldsymbol{\tau_{plane}} - \boldsymbol{\tau_{earth}} = \left(\left(1+\frac{GM}{c^2}\left(\frac{1}{R} - \frac{1}{R+h}\right)\right)\left(1+\frac{({v_{earth}}^2 - {v_2}^2)}{2c^2}\right) - 1\right)\boldsymbol{\tau_{earth}} \qquad (17)$$

$$\boldsymbol{\tau_{plane}} - \boldsymbol{\tau_{earth}} = \left(\left(1+\frac{GM}{c^2}\frac{h}{R^2}\right)\left(1+\frac{({v_{earth}}^2 - {v_2}^2)}{2c^2}\right) - 1\right)\boldsymbol{\tau_{earth}} \qquad (18)$$

$$\boldsymbol{\tau_{plane}} - \boldsymbol{\tau_{earth}} = \left(\frac{GM}{c^2}\frac{h}{R^2} + \frac{(v_{eart\,h}{}^2 - v_2{}^2)}{2c^2}\right)\boldsymbol{\tau_{earth}} \tag{19}$$

$$\boldsymbol{\tau_{plane}} - \boldsymbol{\tau_{earth}} = \left(\frac{gh}{c^2} - \frac{\left(v_2{}^2 - v_{eart\,h}{}^2\right)}{2c^2}\right)\boldsymbol{\tau_{earth}} \tag{20}$$

Note:

The speed of the aircraft is given as the ground speed. It is not immediately clear whether this is measured relative to the clock on Earth or the clock in the aircraft. Let's assume that the clock on Earth is meant. In that case, we have to convert to the level of the aircraft, which means we have to consider the clock at that level. We do this using the time *t* in the universal frame. If we consider $\frac{d\emptyset_{eart\,h}}{dt}$, this is the rotational speed of the Earth in the universal frame. We can find the speed of the Earth at sea level by multiplying $\frac{d\emptyset_{eart\,h}}{dt}$ by R, the distance from the center. The speed of the Earth as seen from the level of the airplane is $(R+h)\frac{d\emptyset_{eart\,h}}{dt}$. The same applies to the airplane: at sea level, the relative airplane speed is $R\frac{d\emptyset_{plane}}{dt}$ and at airplane level $(R+h)\frac{d\emptyset_{plane}}{dt}$.
Now we need to find $\frac{d\emptyset_{eart\,h}}{dt}$ and $\frac{d\emptyset_{plane}}{dt}$.

We use equation_4c from section4.1.5.

$$v_\tau = r\frac{d\emptyset}{d\tau} = r\frac{d\emptyset}{dt}\frac{dt}{d\tau} => \frac{d\emptyset}{dt} = \frac{v_\tau}{r}\frac{d\tau}{dt}$$

Next, we use equation_3_1_5_4b from chapter4.1.5

$$\frac{d\tau}{dt} = \sqrt{\frac{\left(1 - \frac{2GM}{c^2 r}\right)}{1 + \frac{v_\tau{}^2}{c^2}}}$$

So:

$$\frac{d\emptyset}{dt} = \frac{v_\tau}{r}\frac{d\tau}{dt} = \frac{v_\tau}{r}\sqrt{\frac{\left(1 - \frac{2GM}{c^2 r}\right)}{1 + \frac{v_\tau{}^2}{c^2}}}$$

All components on the right-hand side are known.
At sea level:

$$\frac{d\emptyset_{eart\,h}}{dt} = \frac{v_{eart\,h}}{R}\sqrt{\frac{\left(1 - \frac{2GM}{c^2 R}\right)}{1 + \frac{v_{eart\,h}{}^2}{c^2}}}$$

And for the airplane, the same applies:

$$\frac{d\emptyset_{plane}}{dt} = \frac{v_{plane}}{R}\sqrt{\frac{\left(1 - \frac{2GM}{c^2 R}\right)}{1 + \frac{v_{eart\,h}{}^2}{c^2}}}$$

Now at aircraft level:

$$v_2 = v_{2\tau\_eart\,h} + v_{2\tau\_plane} = (R+h)\left(\frac{d\emptyset_{eart\,h}}{dt} + \frac{d\emptyset_{plane}}{dt}\right)$$

$$v_2 = v_{2\tau\_eart\,h} + v_{2\tau\_plane} = \frac{(R+h)}{R}\sqrt{\frac{\left(1-\frac{2GM}{c^2R}\right)}{1+\frac{{v_{eart\,h}}^2}{c^2}}}\left(v_{eart\,h} + v_{plane}\right)$$

Until now, v_2 was still exact.

But now, with a first-order Taylor approximation, it becomes:

$$v_2 = \frac{(R+h)}{R}\sqrt{\left(1-\frac{2GM}{c^2R}\right)\left(1-\frac{{v_{eart\,h}}^2}{c^2}\right)}\left(v_{eart\,h} + v_{plane}\right)$$

So, the relevant formulas are:

$$\boldsymbol{v_2 = \frac{(R+h)}{R}\left(1-\frac{GM}{c^2R}-\frac{{v_{earth}}^2}{2c^2}\right)\left(v_{earth}+v_{plane}\right)}$$

Result after Taylor approximation

Applied to equation (14b), this leads to the following after linearization

$$\boldsymbol{\tau_{plane} - \tau_{earth} = \left(\frac{gh}{c^2} - \frac{\left({v_2}^2 - {v_{earth}}^2\right)}{2c^2}\right)\tau_{earth}} \qquad \mathbf{(20)}$$

Conclusion:

This revised approach corrects the inconsistency in the original derivation: speeds must be related to proper time, not to coordinate time. After correction and Taylor approximation, it appears that the numerical deviation from the approximation in the previous chapter is **less than 0.4%** - within the desired accuracy.

Exact (Formula: 3.1.5.15b):	Paul Anderson	Re_Spec_92	H&K
Total_East	-9.11E-08	-2.35E-07	-4.08E-08
Total_West	3.24E-07	1.81E-07	2.75E-07
sidereal day: 23.9344696hr	86164.1	86164.1	86164.1
Light velocity	299792458	299792458	299792458
G	6.67E-11	6.67E-11	6.67E-11
M_earth	5.97E+24	5.97E+24	5.97E+24
R_earth	6371000	6371000	6371000
Schwarzschild radius Rs:	8.87E-03	8.87E-03	8.87E-03
Formula: 3.1.5.20			
Vplane_ground_east_tau	232.55	670	173.98
Vplane_ground_West_tau	-232.55	-670	-124.43
V_earth_tau	464.58	464.58	464.58
H_east	9000	19000	7664
H_west	9000	19000	9526
t_earth	172328	59747	172328
v2_east	698.12	1137.96	639.33
v2_west	232.36	-206.03	340.56
Grav_delay (ns)_East	1.69E-07	1.24E-07	1.44E-07
Kin_delay (ns)_East	-2.60E-07	-3.59E-07	-1.85E-07
Total_East	-9.09E-08	-2.35E-07	-4.06E-08
Grav_delay (ns)_West	1.69E-07	1.24E-07	1.79E-07
Kin_delay (ns)_West	1.55E-07	5.76E-08	9.57E-08
Total_West	3.25E-07	1.82E-07	2.75E-07
diff east	-2.35E-10	-3.63E-10	-1.56E-10
diff west	-2.18E-10	-3.67E-10	-3.23E-10
diff east in %	0.26	0.15	0.38
diff west in %	-0.07	-0.20	-0.12

4.1.6 Considerations regarding the Hafele-Keating experiment and the Schwarzschild metric

We start with the general Schwarzschild equation:

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\emptyset^2 \tag{1}$$

As used earlier, we define:

$$\sigma = \sqrt{1 - \frac{2GM}{c^2 r}}$$

With this notation, equation (1) is rewritten as:

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \sigma^{-2} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\emptyset^2 \tag{1a}$$

In the Hafele-Keating experiment, the time of the United States Naval Observatory (USNO) clock and the speed of an airplane are measured. The question is: what do time and speed represent in the context of the Schwarzschild metric?

There is a stationary clock at sea level on the equator, and two airplanes moving in the equatorial plane—one to the east, the other to the west. Both airplanes follow a circular path at constant speed relative to the Earth's surface, but in opposite directions.

Since the experiment takes place in the equatorial plane, we assume that $\theta = \frac{\pi}{2}$ is constant, and that *r* is also constant due to the circular orbit. The Schwarzschild metric then simplifies to:

$$c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - r^2 d\emptyset^2 \tag{2}$$

The coordinates $(t, r, \theta, \emptyset)$ in the Schwarzschild metric can be interpreted as belonging to a universal (inertial) reference frame in which the Earth rotates. The clocks on the Earth's surface and in the aircraft are each in their own local inertial frame; their measured time is represented as the proper time τ .

The universal coordination time *t* is not directly measurable, but is a theoretical quantity. From equation (2) it follows that:

$$dt^2 = \frac{d\tau^2 + \frac{r^2}{c^2} d\emptyset^2}{\left(1 - \frac{2GM}{c^2 r}\right)} = \sigma^{-2}\left(d\tau^2 + \frac{r^2}{c^2} d\emptyset^2\right) = \sigma^{-2}\left(1 + \frac{r^2}{c^2}\left(\frac{d\emptyset}{d\tau}\right)^2\right) d\tau^2 \tag{4}$$

Assuming *t=0* when $\tau = 0$, this leads to:

$$t = \sigma^{-1}\sqrt{1 + \frac{r^2}{c^2}\left(\frac{d\emptyset}{d\tau}\right)^2}\,\tau = \sigma^{-1}\sqrt{1 + \frac{v^2}{c^2}}\,\tau \tag{4a}$$

Where $v = r\frac{d\emptyset}{d\tau}$ is the velocity relative to the universal frame. For velocities much smaller than *c*, we can apply a first-order Taylor approximation:

$$\boldsymbol{t} = \sigma^{-1}\sqrt{1+\frac{v^2}{c^2}}\tau = \frac{1}{\sigma\sqrt{1-\frac{v^2}{c^2}}}\tau = \frac{\boldsymbol{\gamma}}{\boldsymbol{\sigma}}\,\boldsymbol{\tau} \qquad (\boldsymbol{4b})$$

with $\gamma = \sqrt{1-\frac{v^2}{c^2}}$ being the Lorentz factor. This equation expresses how the proper timeτ of a moving clock relates to the coordination time *t* in the Schwarzschild system.

4.2 Experiment 2 - Motion of Particles in Schwarzschild Geometry

The derivations in this chapter are largely based on the following sources:

- (Biesel, 2008) *The Precession of Mercury's Perihelion*, Owen Biesel, January 25, 2008 (Biesel, 2008)
- (Magnan) Christian Magnan: *Complete calculations of the perihelion precession of Mercury and the deflection of light by the Sun in General Relativity* (Magnan)
- (Pe'er1, 2014) *Schwarzschild Solution and Black Holes*, Asaf Pe'er1, February 19, 2014 (Pe'er1, 2014)

We will derive equations for the motion of particles in Schwarzschild geometry, which serves as the basis for the following experiments:

- The precession of Mercury's perihelion,
- The deflection of light by the sun,
- The Shapiro experiment
- The calculation of a bullet trajectory.

We use the Schwarzschild metric as a starting point because it exactly satisfies Einstein's field equations and has proven to be widely applicable. Due to the symmetry in both time *t* and the angular coordinate ∅ - that is none of the metric coefficients denpends on *t* or ∅ - the Schwarzschild metric is subject to Noether's theorem. This implies conservation of energy (due to time independence) and conservation of angular momentum (due to ∅ independence).

Schwarzschild metric

The metric is:

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - \frac{r^2}{R_p^2}dR_p^2.\theta^2 - \frac{r^2}{R_p^2}\sin^2\theta\, dR_p^2.\emptyset^2$$

To obtain the correct dimensions (all coordinates in meters), we set *Rp*=1m, which makes the coefficients dimensionless. This leads to the more common form:

$$\boldsymbol{ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta\, d\emptyset^2} \qquad (1a)$$

with:

$$\sigma = \sqrt{1 - \frac{2GM}{c^2 r}} = \sqrt{1 - \frac{R_s}{r}}, \quad \text{where } R_s = \frac{2GM}{c^2}$$

Coefficients of the metric

For the metric in polar coordinates, we find:

$$\boldsymbol{g_{00} = \sigma^2;\ \ g_{11} = \frac{-1}{\sigma^2};\ \ g_{22} = -r^2;\ \ g_{33} = -r^2 \sin^2\theta}$$

For the equatorial plane $\theta = \frac{\pi}{2}$, the following therefore applies $g_{33} = -r^2$.

The contravariant components are:

$$\boldsymbol{g^{00} = \frac{1}{\sigma^2};\ \ g^{11} = -\sigma^2;\ g^{22} = \frac{-1}{r^2};\ \ g^{33} = \frac{-1}{r^2 \sin^2\theta}}$$

Furthermore, the following applies:

$$\frac{d\sigma}{dr} = \frac{R_s}{2r^2\sigma}$$

Derivatives of metric components

$$\boldsymbol{\frac{\partial g_{00}}{\partial r} = \frac{R_s}{r^2};\ \ \frac{\partial g_{11}}{\partial r} = \frac{R_s}{r^2\sigma^4};}$$

$$\boldsymbol{\frac{\partial g_{22}}{\partial r} = -2r;}$$

$$\boldsymbol{\frac{\partial g_{33}}{\partial r} = -2r\sin^2\theta\,;}$$

$$\boldsymbol{\frac{\partial g_{33}}{\partial\theta} = -2r^2 sin\theta \cos\theta}$$

Non-zero Christoffel symbols

Christoffel symbols are given by:

$$\boldsymbol{\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\}}$$

Some relevant non-zero symbols are:

$$\boldsymbol{\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2} g^{00} \left\{ \frac{\partial g_{00}}{\partial r} \right\} = \frac{R_s}{2r^2\sigma^2};\ \ \Gamma^1_{00} = \frac{1}{2} g^{11} \left\{ -\frac{\partial g_{00}}{\partial r} \right\} = \frac{\sigma^2 R_s}{2r^2};\ \ \Gamma^1_{11} = \frac{1}{2} g^{11} \left\{ \frac{\partial g_{11}}{\partial r} \right\} = \frac{-R_s}{2r^2\sigma^2}}$$

$$\boldsymbol{\Gamma^1_{22} = \frac{1}{2} g^{11} \left\{ -\frac{\partial g_{22}}{\partial r} \right\} = -r\sigma^2;\ \ \Gamma^1_{33} = \frac{1}{2} g^{11} \left\{ -\frac{\partial g_{33}}{\partial r} \right\} = -r\sigma^2 \sin^2\theta}$$

$$\boldsymbol{\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2} g^{22} \left\{ \frac{\partial g_{22}}{\partial r} \right\} = \frac{1}{r};\ \ \Gamma^2_{33} = \frac{1}{2} g^{22} \left\{ -\frac{\partial g_{33}}{\partial \theta} \right\} = -\cos\theta\sin\theta}$$

$$\boldsymbol{\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2} g^{33} \left\{ \frac{\partial g_{33}}{\partial r} \right\} = \frac{1}{r};\ \ \Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2} g^{33} \left\{ \frac{\partial g_{33}}{\partial \theta} \right\} = \frac{\cos\theta}{\sin\theta}}$$

All other Christoffel symbols are zero

Geodesic equation and conservation laws

We consider a geodesic world line, which describes the natural trajectory of a particle in the absence of non-gravitational forces. The general form of the geodesic equation is:

$$\frac{d^2x^{\alpha}}{d\lambda^2} + \Gamma^{\alpha}_{\mu\nu}.\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = 0$$

We work out the four coordinates, where λ is the affine parameter (but can be equated here to proper timeτ):

- For *t*:

$$\frac{d^2t}{d\lambda^2} + \Gamma^{t}_{\mu\nu}.\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = \frac{d^2t}{d\lambda^2} + 2\Gamma^0_{01}.\frac{dt}{d\lambda}\frac{dr}{d\lambda} = \frac{d^2t}{d\lambda^2} + 2\frac{R_s}{2r^2\sigma^2}.\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0$$

- For *r*:

$$\frac{d^2r}{d\lambda^2} + \Gamma^{r}_{\mu\nu}.\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = \frac{d^2r}{d\lambda^2} + \Gamma^1_{00}.\left(\frac{dt}{d\lambda}\right)^2 + \Gamma^1_{11}.\left(\frac{dr}{d\lambda}\right)^2 + \Gamma^1_{22}.\left(\frac{d\theta}{d\lambda}\right)^2 + \Gamma^1_{33}.\left(\frac{d\varphi}{d\lambda}\right)^2$$

$$= \frac{d^2r}{d\lambda^2} + \frac{\sigma^2R_s}{2r^2}.\left(\frac{dt}{d\lambda}\right)^2 - \frac{R_s}{2r^2\sigma^2}.\left(\frac{dr}{d\lambda}\right)^2 - r\sigma^2.\left(\frac{d\theta}{d\lambda}\right)^2 - r\sigma^2\sin^2\theta\,.\left(\frac{d\varphi}{d\lambda}\right)^2 = 0$$

- Voor θ:

$$\frac{d^2\theta}{d\lambda^2} + \Gamma^{\theta}_{\mu\nu}.\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = \frac{d^2\theta}{d\lambda^2} + 2\Gamma^2_{12}.\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} + \Gamma^2_{33}.\left(\frac{d\varphi}{d\lambda}\right)^2$$

$$= \frac{d^2\theta}{d\lambda^2} + 2\frac{1}{r}.\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} - \cos\theta\sin\theta\ \ .\left(\frac{d\varphi}{d\lambda}\right)^2 = 0$$

- Voor φ:

$$\frac{d^2\varphi}{d\lambda^2} + \Gamma^{\varphi}_{\mu\nu}.\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = \frac{d^2\varphi}{d\lambda^2} + 2\Gamma^3_{13}.\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda} + 2\Gamma^3_{23}.\frac{d\theta}{d\lambda}\frac{d\varphi}{d\lambda}$$

$$= \frac{d^2\varphi}{d\lambda^2} + 2\frac{1}{r}.\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda} + 2\frac{\cos\theta}{\sin\theta}.\frac{d\theta}{d\lambda}\frac{d\varphi}{d\lambda} = 0$$

In summary, these four component equations lead to:

$$\frac{d^2t}{d\lambda^2} + 2\frac{R_s}{2r^2\sigma^2}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0 \tag{1}$$

$$\frac{d^2r}{d\lambda^2} + \frac{\sigma^2 R_s}{2r^2}\left(\frac{dt}{d\lambda}\right)^2 - \frac{R_s}{2r^2\sigma^2}\left(\frac{dr}{d\lambda}\right)^2 - r\sigma^2\left(\frac{d\theta}{d\lambda}\right)^2 - r\sigma^2\sin^2\theta\left(\frac{d\varphi}{d\lambda}\right)^2 = 0 \tag{2}$$

$$\frac{d^2\theta}{d\lambda^2} + 2\frac{1}{r}\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} - \cos\theta\sin\theta\ \left(\frac{d\varphi}{d\lambda}\right)^2 = 0 \tag{3}$$

$$\frac{d^2\varphi}{d\lambda^2} + 2\frac{1}{r}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda} + 2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{d\lambda}\frac{d\varphi}{d\lambda} = 0 \tag{4}$$

We will first follow the elegant approach of Asaf Pe'er from his article "Schwarzschild Solution and Black Holes" (Pe'er1, 2014) , and then present a simpler approach.

According to Asaf Pe'er:

"At first glance, there does not seem to be much hope of easily solving this set of four coupled equations. Fortunately, our task is greatly simplified by the high degree of symmetry of the Schwarzschild metric."

Schwarzschild space has four Killing fields: three due to spherical symmetry, and one due to time translation. Each Killing field leads, via Noether's theorem, to a constant of motion for a free particle. If K_μ is a Killing field, then:

$$K_\mu \frac{dx^\mu}{d\lambda} = constant. \tag{5}$$

In addition, there is always another constant of motion that follows from metric compatibility. This states that along the geodesic world line, the quantity

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

$$\left(\frac{ds}{d\lambda}\right)^2 = \left(\frac{cd\tau}{d\lambda}\right)^2 = c^2\varepsilon = g_{\mu\nu}\,\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{6}$$

is constant. This is the normalization of the four-velocity. Here, $\varepsilon = 1$ is for massive particles, $\varepsilon = 0$ is for massless particles (such as photons), and $\varepsilon = -1$ is for spatial geodesics.

Instead of directly solving the four coupled geodesic equations, we make use of the symmetries that lead to conservation laws via Killing fields.

In flat space-time, the symmetries, represented by the Killing fields and according to Noether's theorem, lead to well-known conserved quantities:

- Time translation invariance results in energy conservation,
- Rotation invariance in conservation of angular momentum.

For Schwarzschild, the following applies analogously:

- **Motion in a plane**: Angular momentum retains its direction → particle moves in a plane. Due to coordinate rotation, we can choose this as the equatorial plane:

$$\theta = \frac{\pi}{2} \tag{7}$$

- **Conservation of energy**: The time-like Killing field is $K^\mu = (1,0,0,0)^T$, and therefore:

$$K_\mu = K^\nu g_{\mu\nu} = \left(\left(1 - \frac{2GM}{r}\right), 0,0,0\right) \tag{8}$$

It follows that:

$$K_\mu \frac{dx^\mu}{d\lambda} = \left(1 - \frac{2GM}{c^2 r}\right)\frac{dt}{d\lambda} = \frac{E}{c^2}\,, \tag{9}$$

or defined as:

$$k = \left(1 - \frac{2GM}{c^2 r}\right)\frac{dt}{d\lambda} = \frac{E}{c^2} \tag{9a}$$

- **Conservation of angular momentum**: The Killing vector associated with ϕ rotations is $L^\mu = (0,0,0,-1)^T$, and:

$$L_\mu = g_{\mu\nu} L^\nu = (0,0,0,-r^2 \sin^2\theta)\,. \tag{10}$$

At $\theta = \frac{\pi}{2}$, $sin\,\theta = 1$, so that:

$$r^2 \frac{d\phi}{d\lambda} = L \tag{11}$$

This implies the conserved quantities: *E* and *L*, the energy and angular momentum per unit mass. For photons, these are respectively the energy and angular momentum themselves. (For more on angular momentum, see Appendix 10**.**)

Note that equation (11) is the equivalent of Kepler's second law within general relativity: equal areas are traversed in equal times.

Alternative derivation:

Although Asaf Pe'er notes that solving the complete set of coupled geodesic equations seems complex, it turns out that some of these equations are relatively easy to solve. We will demonstrate this using equations (3.2.1) and (3.2.4).

We start with equation (3.2.1):

$$\frac{d^2 t}{d\lambda^2} + 2\frac{R_s}{2r^2\sigma^2}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0$$

Where

$$\sigma^2 = 1 - \frac{2GM}{c^2 r} = 1 - \frac{R_s}{r}$$

We multiply both sides by σ^2 :

$$\sigma^2 \frac{d^2 t}{d\lambda^2} + \frac{R_s}{r^2}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0$$

$$(1 - \frac{R_s}{r})\frac{d^2 t}{d\lambda^2} + \frac{R_s}{r^2}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0$$

We rewrite this as:

$$\frac{d^2 t}{d\lambda^2} + \frac{R_s}{r^2}\frac{dt}{d\lambda}\frac{dr}{d\lambda} - \frac{R_s}{r}\frac{d^2 t}{d\lambda^2} = 0$$

Or:

$$\Rightarrow \frac{d}{d\lambda}\left(\frac{dt}{d\lambda} - \frac{R_s}{r}\frac{dt}{d\lambda}\right) = 0 \ \Rightarrow \frac{d}{d\lambda}\left[\frac{dt}{d\lambda}\left(1 - \frac{R_s}{r}\right)\right] = 0$$

This shows that the expression

$$\frac{dt}{d\lambda}\left(1 - \frac{R_s}{r}\right)$$

is constant along the world line. We recognize this as the conserved quantity related to the total energy per unit mass of the particle. Multiplying by *c* gives:

$$\boldsymbol{\frac{cdt}{d\lambda}\left(1 - \frac{R_s}{r}\right) = constant = \frac{E}{c} \ \ (total\ energy)} \qquad \mathbf{(9)}$$

We now continue with equation (3.2.4). To simplify the derivation, we assume that the particle moves in the equatorial plane, so th $\theta = \frac{\pi}{2}$. Then:

$$\frac{d^2\varphi}{d\lambda^2} + 2\frac{1}{r}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda} + 2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{d\lambda}\frac{d\varphi}{d\lambda} = 0$$

$$\frac{d^2\varphi}{d\lambda^2} + 2\frac{1}{r}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda} = 0$$

We rewrite this as:

$$\frac{1}{r^2}\frac{d}{d\lambda}\left(r^2\frac{d\varphi}{d\lambda}\right) = 0$$

This also means that:

$$r^2\frac{d\varphi}{d\lambda}$$

is a constant along the geodesic world line. We recognize this constant as the angular momentum per unit mass:

$$\boldsymbol{r^2\frac{d\varphi}{d\lambda} = constant = L\ (impulsmoment)} \qquad \mathbf{(11)}$$

Due to the symmetries, there are two conserved quantities:

- **Energy per unit mass:**

$$\left(\boldsymbol{1}-\frac{\boldsymbol{R_s}}{\boldsymbol{r}}\right)\frac{\boldsymbol{dt}}{\boldsymbol{d\lambda}}=\frac{\boldsymbol{E}}{\boldsymbol{c^2}}$$

- **Angular momentum per unit mass**:

$$\boldsymbol{L}=\boldsymbol{r^2}\frac{\boldsymbol{d\varphi}}{\boldsymbol{d\lambda}}$$

4.2.1 The gravitational potential

Using the conservation laws derived earlier, we can now further analyze the motion of particles in the Schwarzschild metric. We start by writing out equation (6), using the conserved quantities from equations (10) and (11):

$$\left(1-\frac{2GM}{c^2r}\right)c^2\left(\frac{dt}{d\lambda}\right)^2-\left(1-\frac{2GM}{c^2r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2-r^2\left(\frac{d\phi}{d\lambda}\right)^2=c^2\varepsilon. \tag{12}$$

Substituting the conserved quantities E and L gives:

$$\left(1-\frac{2GM}{c^2r}\right)c^2\left(\frac{dt}{d\lambda}\right)^2-\left(1-\frac{2GM}{c^2r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2-\frac{L^2}{r^2}=c^2\varepsilon$$

We multiply this equation by $\left(1-\frac{2GM}{c^2r}\right)$ and use $E/c=c\frac{dt}{d\lambda}\left(1-\frac{2GM}{c^2r}\right)$ and $L=r^2\frac{d\phi}{d\lambda}$ to rewrite:

$$\left(1-\frac{2GM}{c^2r}\right)^2c^2\left(\frac{dt}{d\lambda}\right)^2-\left(\frac{dr}{d\lambda}\right)^2-\left(1-\frac{2GM}{c^2r}\right)\left(\frac{L^2}{r^2}+c^2\varepsilon\right)=0$$

Substituting the expression for E leads to:

$$\frac{E^2}{c^2}-\left(\frac{dr}{d\lambda}\right)^2-\left(1-\frac{2GM}{c^2r}\right)\left(\frac{L^2}{r^2}+c^2\varepsilon\right)=0\,. \tag{13}$$

This has enabled us to reduce the four coupled geodesic equations to a single differential equation for $r(\lambda)$. This greatly simplifies the problem.

We rewrite equation (13) in the following form:

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2+V(r)=\frac{1}{2}\frac{E^2}{c^2}\,. \tag{14}$$

with the effective potential *V(r)* defined as:

$$V(r)=\frac{1}{2}c^2\varepsilon-\varepsilon\frac{GM}{r}+\frac{L^2}{2r^2}-\frac{GML^2}{c^2r^3}\,. \tag{15}$$

Equation (14) is formally identical to the classical equation for the motion of a particle (with unit mass) in a one-dimensional potential *V (r)*, where the total "energy" is $\frac{1}{2}\frac{E^2}{c^2}$. Of course, the actual energy is *E*, but this form makes the equation analogous to classical mechanics. The first term represents the kinetic energy, the second the potential, and their sum is constant.

If we analyze the potential *V(r)* in equation (15), we see that it differs from the Newtonian potential in only one respect: the last term. This term, which is proportional to $1/r^3$, represents a purely relativistic correction and plays an important role especially for small *r*.

The terms can be interpreted as follows:

- The first term is constant (rest energy for massive particles) (dependent on $\varepsilon = 1$ for massive particles and $\varepsilon = 0$ for photons),
- The second term is the Newtonian gravitational potential,
- The third term is the classical angular momentum potential,
- The fourth term is the relativistic correction.

Note: despite the formal similarity to classical mechanics, this does not describe a particle moving freely in one dimension. In reality, this concerns a particle moving in orbit around a massive object. The quantities of interest are *r(λ)*, but also *t(λ)* and $\phi(\lambda)$, which together describe the space-time trajectory (see Figure 1).

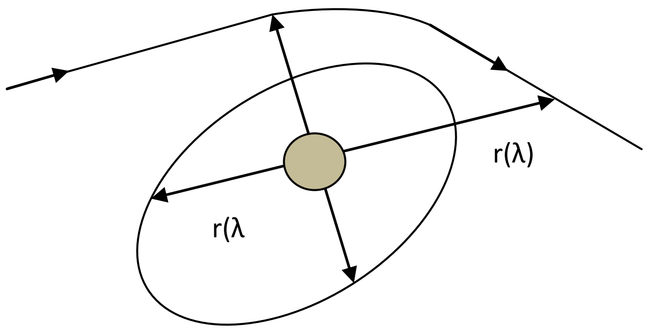


Figure 1 - Trajectories of particles in a gravitational potential.

4.2.2 Interlude on Energy in Schwarzschild Geometry

In this interlude, we analyze the form of energy as derived in equation (3.2.9) of section (4.2). This energy is a conserved quantity in Schwarzschild geometry.

We start with the relation:

$$\left(1 - \frac{2GM}{c^2 r}\right)\frac{dt}{d\lambda} = \frac{E}{mc^2} = \sigma^2\ \frac{dt}{d\lambda}, \tag{9}$$

From which it follows:

$$E = \sigma^2 mc^2\ \frac{dt}{d\lambda}$$

The Schwarzschild metric is:

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta\ d\emptyset^2$$

To avoid the situation around the origin $\tau = 0$, we use an affine parameter λ , with $d\tau = d\lambda$. We then restrict ourselves to the equatorial plane $(\theta = \pi/2)$, which simplifies the metric.

The norm of the 4-velocity then gives:

$$\sigma^2 c^2 \left(\frac{dt}{d\lambda}\right)^2 - \sigma^{-2} \left(\frac{dr}{d\lambda}\right)^2 - r^2 \left(\frac{d\phi}{d\lambda}\right)^2 = c^2 \varepsilon.$$

By rewriting to velocities relative to the coordinate time t, we obtain:

$$\sigma^2 c^2 \left(\frac{dt}{d\lambda}\right)^2 - \sigma^{-2} \left(\frac{dr}{dt}\right)^2 \left(\frac{dt}{d\lambda}\right)^2 - r^2 \left(\frac{d\phi}{dt}\right)^2 \left(\frac{dt}{d\lambda}\right)^2 = c^2 \varepsilon$$

For particles with mass, $\varepsilon = 1$:

$$\sigma^2 \left(\frac{dt}{d\lambda}\right)^2 \left(1 - \frac{\sigma^{-2} \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\phi}{dt}\right)^2}{\sigma^2 c^2}\right) = \varepsilon = 1 \tag{9a}$$

$$\left(\frac{dt}{d\lambda}\right)^2 \left(1 - \frac{v^2}{\sigma^2 c^2}\right) = \frac{1}{\sigma^2}$$

From which it follows:

$$\frac{dt}{d\lambda} = \frac{1}{\sigma\sqrt{\left(1 - \frac{v^2}{\sigma^2 c^2}\right)}}$$

Where $v^2 = \sigma^{-2} \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\phi}{dt}\right)^2$ is the total velocity. In combination with the energy (equation 9), this leads to:

$$E = \sigma^2 mc^2 \frac{dt}{d\lambda}$$

So:

$$\boldsymbol{E = \frac{\sigma m c^2}{\sqrt{\left(1 - \frac{v^2}{\sigma^2 c^2}\right)}}}$$

$$\boldsymbol{E = \gamma_\sigma \sigma m c^2}$$

This expression gives the total conserved energy. From this we can distinguish three components:

- **Rest energy:** $E = \sigma mc^2$
- **Relativistic kinetic energy:**

$$E_{kin} = \sigma mc^2 \left[\frac{1}{\sqrt{\left(1 - \frac{v^2}{\sigma^2 c^2}\right)}} - 1\right]$$

In the non-relativistic limit $v \ll c$, and with a first-order Taylor expansion of the root, we find the "kinetic" energy:

$$\boldsymbol{E_{kin}} \approx \sigma mc^2 \left[1 + \frac{v^2}{2\sigma^2 c^2} - 1\right] = \frac{\boldsymbol{mv^2}}{\boldsymbol{2\sigma}}$$

4.2.2.1 Alternative approach via the metric

Start with the Schwarzschild equation:

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta \, d\emptyset^2$$

$$\sigma^2 c^2 \left(\frac{dt}{d\lambda}\right)^2 - \sigma^{-2}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2 = c^2\varepsilon$$

Multiply by σ^2 :

$$\sigma^4 c^2 \left(\frac{dt}{d\lambda}\right)^2 - \left(\frac{dr}{d\lambda}\right)^2 - \sigma^2 r^2\left(\frac{d\phi}{d\lambda}\right)^2 = \sigma^2 c^2\varepsilon$$

Use again:

$$E = \sigma^2 mc^2 \frac{dt}{d\lambda} => \frac{E}{mc} = \sigma^2 c \frac{dt}{d\lambda}$$

And write:

$$\left(\frac{E}{mc}\right)^2 = \left(\frac{dr}{d\lambda}\right)^2 + \sigma^2 r^2\left(\frac{d\phi}{d\lambda}\right)^2 + \sigma^2 c^2\varepsilon$$

Suppose

$$r^2 \frac{d\phi}{d\lambda} = \frac{L}{m} => r^2\left(\frac{d\phi}{d\lambda}\right)^2 = \frac{L^2}{r^2 m^2} = \frac{(mv_t r)^2}{r^2 m^2} = {v_t}^2$$

Now we take $\lambda = \tau$ en $\varepsilon = 1$:

$$\left(\frac{E}{mc}\right)^2 = \left(\frac{dr}{d\tau}\right)^2 + \sigma^2 {v_t}^2 + \sigma^2 c^2 = {v_r}^2 + \sigma^2 {v_t}^2 + \sigma^2 c^2$$

$$\left(\frac{E}{c}\right)^2 = \mathrm{m}^2 {v_r}^2 + \mathrm{m}^2\sigma^2 {v_t}^2 + \mathrm{m}^2\sigma^2 c^2$$

Where

$$v_r = \frac{dr}{d\tau}, \text{and } v_t = \mathrm{r}\frac{d\phi}{d\tau}$$

We interpret the terms in this expression as follows:

- mv_r: radiale impuls
- $m\sigma v_t$: transversale impuls
- σmc^2: rustenergie

The kinetic energy is then:

$$\boldsymbol{E_{kin}} = \boldsymbol{m}c\sqrt{{v_r}^2 + \sigma^2 {v_t}^2}$$

4.2.2.2 Third approach: via a relativistic energy-momentum relation

Start again with the 4-velocity norm:

$$\sigma^2 c^2 \left(\frac{dt}{d\lambda}\right)^2 - \sigma^{-2}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2 = c^2\varepsilon$$

From which it follows:

$$\left(\frac{E}{\sigma c}\right)^2 - \sigma^{-2}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2 = c^2\varepsilon$$

$$\left(\frac{E}{\sigma c}\right)^2 - p^2 = c^2\varepsilon => \left(\frac{E}{\sigma c}\right)^2 = c^2\varepsilon + p^2$$

Write this as:

$$E^2 = \sigma^2 c^4 \varepsilon + \sigma^2 p^2 c^2$$

where *p* is the total spatial momentum per unit mass. At rest, $E = \sigma m c^2$, for a photon $E = \sigma p c$, and in general

Or:

$$E^2 = \sigma^2 m^2 c^4 \varepsilon + \sigma^2 m^2 c^2 U^2$$

where

$$U^2 = (dx^\mu/d\tau)^2 = \sigma^{-2}(dr/d\tau)^2 + r^2(d\phi/d\tau)^2$$

is the quadratic norm of the spatial velocity. For massive particles ($\varepsilon = 1$) and unit mass, this becomes:

$$E^2 = \sigma^2 c^4 + \sigma^2 c^2 U^2$$

4.2.3 Summary

- The Schwarzschild metric and the geodesic equations derived from it form the basis for explaining a wide range of experiments in general relativity.
- By using symmetries and conserved quantities (Noether's theorem), the complex equations of motion are reduced to manageable forms.
- The effective potential contains all classical and relativistic effects measured in experiments such as the precession of Mercury, light deflection, Shapiro time delay, and bullet trajectories.

4.3 Experiment 3 - Deflection of Light

4.3.1 Historical and theoretical background

The deflection of light by gravity was the first experimental test of the general theory of relativity. In classical Newtonian gravity, light, as a massless phenomenon, moves in straight lines that are not affected by gravity. According to general relativity, however, light follows the curvature of space-time caused by mass. As a result, a light ray deviates from a straight line when it passes by a massive object such as the sun.

This effect can be observed when we look at the light from a star that is visually close to the sun. When the light from the star grazes the sun, it is deflected, so that the star appears to be in a different place in the sky than where it actually is. Six months later, when the star is on the other side of the sky, its light will pass the sun at a great distance, and its position will be observed correctly.

To make this effect visible, a solar eclipse is necessary, because otherwise the sunlight outshines the starlight. In 1919, this effect was measured for the first time by Arthur Eddington during a total solar eclipse. His observations confirmed Einstein's prediction and represented a major breakthrough in the acceptance of the general theory of relativity.

4.3.2 The derivation of the deflection angle

Consider a light ray (photon) approaching from infinity and moving past the sun. The motion of the photon in Schwarzschild space is described by the effective energy equation, as derived in section4.2.1(equations 3.2.1.14 and 3.2.1.15). For a photon, ε = 0, so that:

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r\,) = \frac{1}{2}\frac{E^2}{c^2}\,. \tag{14}$$

Together with:

$$V(r\,) = \frac{1}{2}c^2\varepsilon - \varepsilon\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{c^2r^3} \tag{15}$$

Where L is the angular momentum and E is the energy of the photon.

Inserting this into (14) and knowing that for a photon ε = 0, we get:

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{2r^2} - \frac{GML^2}{c^2r^3} = \frac{1}{2}\frac{E^2}{c^2}\,.$$

We divide this expression by L^2 and multiply by 2:

$$\frac{1}{L^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{r^2} - \frac{2GM}{c^2r^3} = \frac{E^2}{c^2L^2}\,.$$

$$\frac{1}{L^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{r^2}\left(1 - \frac{2GM}{c^2r}\right) = \frac{E^2}{c^2L^2}$$

Isolating $(dr/d\lambda)^2$ yields:

$$\left(\frac{dr}{d\lambda}\right)^2 = L^2\left[\frac{E^2}{c^2L^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{c^2r}\right)\right] \tag{16}$$

4.3.3 Impact parameter and angular momentum

The **impact parameter** *b* is the distance between the center of mass of the massive object (the sun) and the asymptotic direction of the light ray at infinity (see Figure 2).

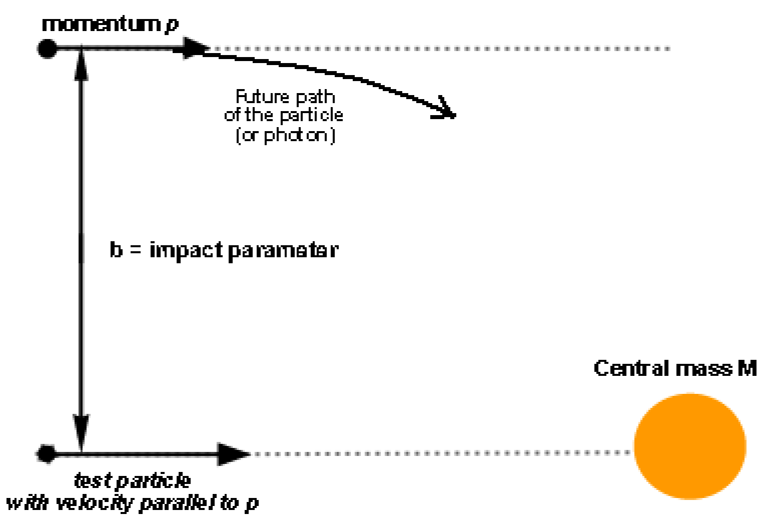


Figure 2. Definition of the impact parameter *b*. The moving particle approaches the mass M from a great distance with vector momentum *p*. A test particle with a parallel velocity dives radially toward the mass M. The distance *b* between their initially parallel paths in the "infinity" is the impact parameter *b*.

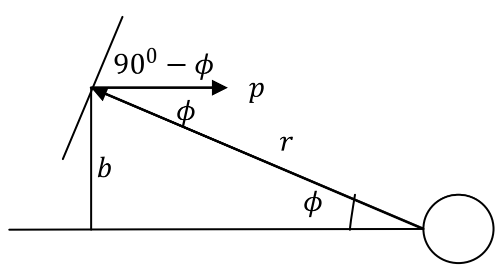
The **angular momentum** *L* of the photon is then:

$$L = pb \tag{17}$$

The energy in general is $E^2 = p^2c^2 + m^2c^4$; , while for a photon *m=0*, so *E=pc*, such that:

$$b = \frac{L}{E/c} \tag{18}$$

Further clarification of the relationship (17) and (18):



The angular momentum is $L = p\sin\phi \cdot r = p \cdot r\sin\phi = p \cdot b$

The energy in general is $E^2 = p^2c^2 + m^2c^4$; and for a photon *m=0*, so *E=pc*.
Therefore:

$$\frac{L}{E/c} = \frac{pb}{pc/c} = b$$

From this we can also deduce that:

$$\frac{1}{b^2} = \frac{E^2}{c^2L^2} \tag{18a}$$

4.3.4 Derivation of the path: the trajectory equation for the photon

Using equation equation_4_2_11, $\left(r^2\frac{d\phi}{d\lambda} = L\right)$, we find:

$$\frac{d\phi}{d\lambda} = \frac{d\phi}{dr}\frac{dr}{d\lambda} = \frac{L}{r^2} \quad => \frac{d\phi}{dr} = \frac{L}{r^2}\left(\frac{dr}{d\lambda}\right)^{-1}$$

Together with (16), this gives:

$$\frac{d\phi}{dr} = \frac{L}{r^2}\left(\frac{dr}{d\lambda}\right)^{-1} = \pm\frac{L}{r^2}\frac{1}{L}\left[\frac{E^2}{c^2L^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{c^2r}\right)\right]^{-1/2}$$

With (18a), this becomes:

$$\frac{d\phi}{dr} = \pm\frac{1}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{c^2r}\right)\right]^{-1/2} \tag{19}$$

This leads to the differential equation:

$$\left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2 = \frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{c^2r}\right) \tag{20}$$

4.3.5 Integration of the path

The angle of deflection is obtained by calculating the change in angle $\Delta\phi$ along the path of the photon, from infinity to the perihelion *r=R*, and back again. From (19) we get:
(See Figure 3).

$$\Delta\phi = 2\int_{r_1}^{\infty}\frac{dr}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{c^2r}\right)\right]^{-\frac{1}{2}} \tag{21}$$

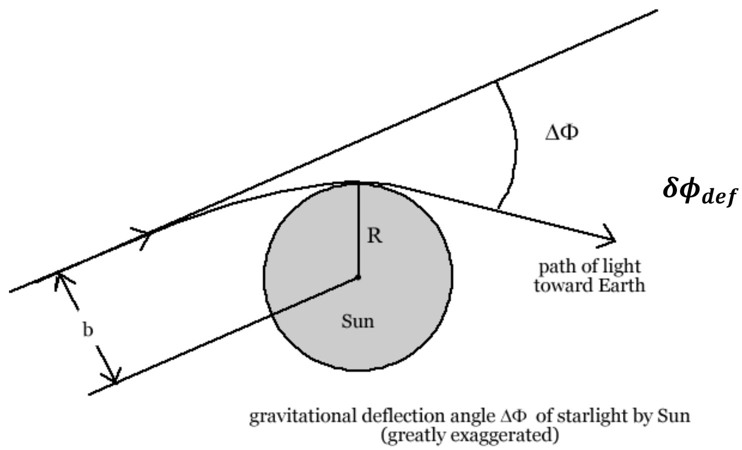


Fig. 3- Deflection of light by angle $\delta\varphi_{def}$

At the turning point *r = R*, the following applies $dr/d\phi = 0$, so that (20) gives:

$$\frac{1}{b^2} = \frac{1}{R^2}\left(1 - \frac{2GM}{c^2 R}\right) \tag{22}$$

Substitute this into (20):

$$\left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2 = \frac{1}{R^2}\left(1 - \frac{2GM}{c^2 R}\right) - \frac{1}{r^2}\left(1 - \frac{2GM}{c^2 r}\right) \tag{23}$$

4.3.6 Variable substitution

Perform the following substitution:

$$u = \frac{R}{r}$$

$$\frac{du}{d\phi} = \frac{du}{dr}\frac{dr}{d\phi} = \frac{-R}{r^2}\frac{dr}{d\phi} \Rightarrow \left(\frac{du}{d\phi}\right)^2 = \left(\frac{R}{r^2}\frac{dr}{d\phi}\right)^2$$

Where ***u*** vary between 1 (r=R) and 0 (r=∞). Equation (23) then becomes:

$$\left(\frac{du}{d\phi}\right)^2 = \left(1 - \frac{2GM}{c^2 R}\right) - u^2\left(1 - \frac{2GMu}{c^2 R}\right)$$

Or:

$$\left(\frac{du}{d\phi}\right)^2 = 1 - u^2 - \frac{2GM}{c^2 R}(1 - u^3) \tag{24}$$

From this it follows:

$$d\phi = \left[1 - u^2 - \frac{2GM}{c^2 R}(1 - u^3)\right]^{-\frac{1}{2}} du$$

$$= \frac{(1-u^2)^{-1/2}}{\left[1-\frac{2GM}{c^2R}(1-u^3)(1-u^2)^{-1}\right]^{\frac{1}{2}}}du \qquad (25)$$

This integral is difficult to solve in closed form. To simplify it, we use the substitution:

$$\mathrm{u} = \cos\alpha \text{ with } \; 0 < \alpha < \pi/2 \text{ so } 0 < u < 1$$

Then:

$$d\phi = -\left[1-\frac{2GM}{c^2R}(1-\cos^3\alpha)\sin^{-2}\alpha\right]^{-\frac{1}{2}} d\alpha \qquad (26)$$

By recognizing that:

$$\frac{1-\cos^3\alpha}{\sin^2\alpha} = \frac{(1-\cos\alpha)(1+\cos\alpha+\cos^2\alpha)}{(1-\cos\alpha)(1+\cos\alpha)} = \frac{1+\cos\alpha\,(1+\cos\alpha)}{(1+\cos\alpha)} = \cos\alpha + \frac{1}{(1+\cos\alpha)}$$

We ultimately obtain:

$$d\phi = -\left[1-\frac{2GM}{c^2R}\left(\cos\alpha + \frac{1}{(1+\cos\alpha)}\right)\right]^{-\frac{1}{2}} d\alpha \qquad (27)$$

With:

$$\cos\alpha = R/r$$

Up to this point**, no approximation** has been applied. This complete derivation is suitable for calculating the light deflection exactly, although in practice a first-order approximation is often sufficient to determine the angle of deflection along the edge of the sun.

4.3.6.1 Approximations and Integration.

The value of the parameter

$$\frac{2GM}{c^2R} \approx 4.24 \cdot 10^{-6}$$

is very small at the surface of the sun. This allows us to use an approximation to solve the integral in equation (27).

We apply the well-known Taylor approximation:

$$\frac{1}{\sqrt{1-k}} \approx 1 + \frac{1}{2}k \qquad \text{for small } k$$

Applied to equation (27), this yields:

$$d\phi = -\left[1-\frac{2GM}{c^2R}\left(\cos\alpha + \frac{1}{(1+\cos\alpha)}\right)\right]^{-\frac{1}{2}} d\alpha$$

Or:

$$d\phi = -\frac{1}{\sqrt{\left[1-\frac{2GM}{c^2R}\left(\cos\alpha + \frac{1}{(1+\cos\alpha)}\right)\right]}} d\alpha$$

After approximation:

$$d\phi \approx -\left[1+\frac{GM}{c^2R}\left(\cos\alpha+\frac{1}{(1+\cos\alpha)}\right)\right]d\alpha \tag{28}$$

We can now calculate the total change in azimuth along the entire path of the photon, from $\alpha = 0$ to $\alpha = \frac{\pi}{2}$, and double this:

$$\Delta\phi = 2\int_0^{\frac{\pi}{2}}\left[1+\frac{GM}{c^2R}\left(\cos\alpha+\frac{1}{(1+\cos\alpha)}\right)\right]d\alpha \tag{29}$$

To work out this integral, we look at the second term separately:

To calculate the integral

$$\int\frac{1}{1+\cos\alpha}d\alpha$$

We use the trigonometric identity:

$$1+\cos\alpha = 1+\cos\left(\frac{\alpha}{2}+\frac{\alpha}{2}\right) = \cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2} = 2\cos^2\frac{\alpha}{2}$$

So:

$$\frac{1}{1+\cos\alpha} = \frac{1}{2\cos^2\frac{\alpha}{2}}$$

Note that:

$$\frac{1}{2\cos^2\frac{\alpha}{2}} = \frac{1}{2}\left(1+\frac{\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}\right) = \frac{1}{2}\left(\frac{\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}+\frac{\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}\right) = \frac{1}{2}\frac{\mathrm{d}\left(\tan\frac{\alpha}{2}\right)}{\mathrm{d}\left(\frac{\alpha}{2}\right)} = \frac{\mathrm{d}\left(\tan\frac{\alpha}{2}\right)}{\mathrm{d}\alpha}$$

So

$$\int\frac{1}{1+\cos\alpha}d\alpha = \tan\frac{\alpha}{2}$$

Now we can evaluate the complete integral (29):

$$\Delta\phi = 2\left[\alpha+\frac{GM}{c^2R}\left(\sin\alpha+\tan\frac{\alpha}{2}\right)\right]_0^{\frac{\pi}{2}} \tag{30}$$

We substitute:

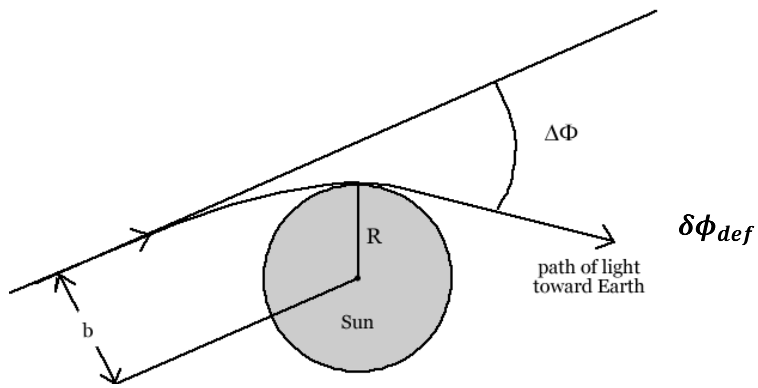
- $\alpha = \frac{\pi}{2}$: $\sin\left(\frac{\pi}{2}\right) = 1, \tan\left(\frac{\pi}{4}\right) = 1$
- $\alpha = 0$: $\sin(0) = 0, \tan(0) = 0$

So:

$$\Delta\phi = 2\left[\frac{\pi}{2}+\frac{GM}{c^2R}(1+1)\right] = \pi+\frac{4GM}{c^2R}$$

$$\Delta\phi = \pi+\frac{4GM}{c^2R} \tag{31}$$

Note: the integral should go from $r = \infty$ to R, so now *you* go from 0 to 1, and thus α van $\frac{\pi}{2}$ naar 0 . By changing the integral to 0 till $\frac{\pi}{2}$, the sign changes and the minus sign disappears.



$\boldsymbol{gravitational\ deflection\ angle\ \delta\phi_{def}\ of\ the\ star\ light\ by\ the\ Sun}$

The first term, π, is the total angle change of a photon in flat space-time—a straight path without deflection. The second term is the extra deflection due to the curvature of space-time. The actual angle of deflection is therefore:

$$\delta\phi_{def} = \Delta\phi - \pi \approx \frac{4GM}{c^2 R} \tag{32}$$

Numerical value

With:

- $G = 6.674 \cdot 10^{-11} \mathrm{Nm^2/kg^2}$
- $M = 1.989 \cdot 10^{30} \mathrm{kg}$
- $\mathrm{c} = 3.00 \cdot 10^{8}\ m/s$
- $R = 6.963 \cdot 10^{8} \mathrm{m}$

we find:

$$\delta\phi_{def} = \frac{4 \cdot 6.674 \cdot 10^{-11} \cdot 1.989 \cdot 10^{30}}{(3.00 \cdot 10^{8}) \cdot 6.963 \cdot 10^{8}} \approx 8.5 \cdot 10^{-6} \text{ radians}$$

To convert this into arc seconds, we use:

$$1 \text{ rad} = \frac{180 \cdot 60 \cdot 60}{\pi} \approx 206.265''$$

From this it follows:

$$\delta\phi_{def} \approx 8.5 \cdot 10^{-6} \cdot 206.265 \approx 1.75''$$

4.3.7 Conclusion

This deflection of 1.75 arc seconds was first observed by Arthur Eddington during the solar eclipse of 1919. The result spectacularly confirmed Einstein's prediction and marked a milestone in the experimental confirmation of the general theory of relativity.

This effect can also be seen outside our solar system and is known as "gravitational lensing."

4.3.8 Physical Interpretation

- **Light follows the curvature of space-time:**
  Near a mass such as the sun, the path of light is deflected; the star appears to be in a different place in the sky.
- **No force on the photon:**
  The deflection is a purely geometric effect, not the result of a force acting on the light particle.
- **Practical effect:**
  Visible during a solar eclipse, when stars close to the sun appear to shift in the sky.

4.3.9 Key insights

- The deflection of light is a direct consequence of the curvature of space-time around mass.
- The Schwarzschild metric provides a quantitative prediction that has been confirmed experimentally.
- This experiment was crucial for the early acceptance of general relativity.

4.4 Experiment 4 - Precession of the Perihelia (Mercury)

Based on an article by Owen Biesel (Biesel, 2008).

4.4.1 Introduction

- **Physical problem:**
  Mercury's orbit around the sun is an ellipse, but the closest point (perihelion) slowly shifts over time. This phenomenon is called precession of the perihelion.
- **Classical explanation:**
  Newtonian mechanics explains most of this precession (due to the influence of other planets), but a residue of approximately 43 arc seconds per century remains unexplained.

- **Relativistic explanation:**
  General relativity predicts an additional precession due to the curvature of space-time around the sun, corresponding exactly to the observed surplus.

4.4.2 Theoretical framework: Schwarzschild metric

In general relativity, we consider a planet such as Mercury as a test particle moving along a geodesic path through curved space-time.

- The Schwarzschild metric describes this space-time around a spherically symmetric mass (such as the sun):

$$\boldsymbol{ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2} \qquad (\mathbf{33})$$

with:

$$\sigma = \sqrt{1 - \frac{2GM}{c^2 r}} = \sqrt{1 - \frac{R_s}{r}}, \quad \text{where } R_s = \frac{2GM}{c^2}$$

- For a planet moving in the equatorial plane$\left(\theta = \frac{\pi}{2}\right)$, this simplifies to:

$$1 = \left(1 - \frac{R_s}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{c^2}\left(1 - \frac{R_s}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - \frac{1}{c^2} r^2 \left(\frac{d\emptyset}{d\tau}\right)^2$$

4.4.3 Derivation of the precession via the Lagrange approach (see Appendix 12)

Although we have already derived the expressions for energy *E* (equation 3.2 9) and angular momentum *L* (equation 3.2 11), we will repeat the derivation from the Lagrangian here. We parameterize the orbit as $x^\mu(\tau) = \big(t(\tau), r(\tau), \theta(\tau), \emptyset(\tau)\big)$ withτ the proper time. In the equatorial plane $(\theta = \pi/2)$, the Lagrangian becomes:

$$\mathcal{L} = \left(1 - \frac{R_s}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{c^2}\left(1 - \frac{R_s}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - \frac{1}{c^2} r^2 \left(\frac{d\emptyset}{d\tau}\right)^2 \qquad (34)$$

The Euler-Lagrange equations for Ø and *t* give:

$$\frac{d\emptyset}{d\tau} = \dot{\emptyset} \text{ and } \frac{dt}{d\tau} = \dot{t}$$

$$\mathcal{L} = \left(1 - \frac{R_s}{r}\right)\dot{t}^{\,2} - \frac{1}{c^2}\left(1 - \frac{R_s}{r}\right)^{-1}\dot{r}^2 - \frac{1}{c^2} r^2 \dot{\emptyset}^2$$

4.4.4 Euler-Lagrange operation:

$$\text{here is for } \emptyset\text{: } \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{\emptyset}}\right) = \frac{\partial \mathcal{L}}{\partial \emptyset} = 0$$

$$\text{and for } t\text{: } \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) = \frac{\partial \mathcal{L}}{\partial t} = 0$$

Then the Euler-Lagrange equations for∅ , and t are:

$$0 = \frac{d}{d\tau}\left(2\frac{1}{c^2}r^2\frac{d\emptyset}{d\tau}\right) \;\;=>\;\; r^2\frac{d\emptyset}{d\tau} = \text{constant} = L$$

$$0 = \frac{d}{d\tau}\left(2\left(1-\frac{R_s}{r}\right)\frac{dt}{d\tau}\right) \;\;=>\;\; \left(1-\frac{R_s}{r}\right)\frac{dt}{d\tau} = \text{constant} = \frac{E}{c^2}$$

It follows that L (the angular momentum per unit mass) and E (the energy per unit mass) are constants of motion.
We rewrite the original norm condition:

$$1 = \left(1-\frac{R_s}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{c^2}\left(1-\frac{R_s}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - \frac{1}{c^2}r^2\left(\frac{d\emptyset}{d\tau}\right)^2$$

and substitute the constants E and L into it:

$$1 = \frac{\frac{E^2}{c^4}}{1-\frac{R_s}{r}} - \frac{1}{c^2}\frac{\left(\frac{dr}{d\tau}\right)^2}{1-\frac{R_s}{r}} - \frac{1}{c^2}\frac{L^2}{r^2}$$

$$1-\frac{R_s}{r} = \frac{E^2}{c^4} - \frac{1}{c^2}\left(\frac{dr}{d\tau}\right)^2 - \frac{1}{c^2}\frac{L^2}{r^2} + \frac{1}{c^2}\frac{L^2}{r^2}\frac{R_s}{r}$$

$$\left(\frac{dr}{d\tau}\right)^2 = c^2\left(\frac{E^2}{c^4}-1\right) + c^2\frac{R_s}{r} - \frac{L^2}{r^2} + \frac{R_sL^2}{r^3}$$

By:

$$\frac{dr}{d\tau} = \frac{dr}{d\emptyset}\frac{d\emptyset}{d\tau} = \frac{dr}{d\emptyset}\frac{L}{r^2} => \left(\frac{dr}{d\emptyset}\right)^2 = \frac{r^4}{L^2}\left(\frac{dr}{d\tau}\right)^2$$

We obtain:

$$\left(\frac{dr}{d\emptyset}\right)^2 = \frac{r^4}{L^2}\left[c^2\left(\frac{E^2}{c^4}-1\right) + \frac{c^2R_s}{r} - \frac{L^2}{r^2} + \frac{R_sL^2}{r^3}\right]$$

$$\left(\frac{dr}{d\emptyset}\right)^2 = \text{Newtonian terms} + \text{relativistic correction}$$

The extra $\frac{R_sL^2}{r^3}$ term in the potential causes the precession of the perihelion.
After simplification:

$$\left(\frac{dr}{d\emptyset}\right)^2 = \frac{c^2}{L^2}\left(\frac{E^2}{c^4}-1\right)r^4 + \frac{c^2R_s}{L^2}r^3 - r^2 + R_sr$$

From this we derive:

$$d\emptyset = \frac{1}{\sqrt{\frac{c^2}{L^2}\left(\frac{E^2}{c^4}-1\right)r^4 + \frac{c^2R_s}{L^2}r^3 - r^2 + R_sr}}\,dr$$

4.4.5 Precession of the orbit

For a closed orbit, the radial motion must be limited, i.e. $dr/(d\emptyset) = 0$ at two points: the perihelion *P* and the aphelion *A*. The angular displacement between these two points is:

$$\emptyset_{\boldsymbol{A}} - \emptyset_{\boldsymbol{P}} = \int_P^A \frac{dr}{\sqrt{\frac{c^2}{L^2}\left(\frac{E^2}{c^4} - 1\right)r^4 + \frac{c^2 R_s}{L^2}r^3 - r^2 + R_s r}} \tag{35}$$

To express *E* and *L* in terms of *A*, *P*, and R_s , we set $dr/(d\emptyset) = 0$ for *r*= *A* and *r*= *P.* This leads to the following equations:

$$c^2\left(\frac{E^2}{c^4} - 1\right)A^4 + (L^2)(-A^2 + R_s A) = -c^2 R_s A^3 \tag{36}$$

$$c^2\left(\frac{E^2}{c^4} - 1\right)P^4 + (L^2)(-P^2 + R_s P) = -c^2 R_s P^3 \tag{37}$$

By suitable combinations and subtractions of these equations, we can express $\frac{E^2}{c^4} - 1$ and L^2 entirely in terms of *A*, *P*, and R_s . (See original derivation for details.)

Multiply (36) by $(-P^2 + R_s P)$

$$c^2\left(\frac{E^2}{c^4} - 1\right)A^4(-P^2 + R_s P) + (L^2)(-A^2 + R_s A)(-P^2 + R_s P) = -c^2 R_s A^3(-P^2 + R_s P)$$

Multiply (37) by $(-A^2 + R_s A)$

$$c^2\left(\frac{E^2}{c^4} - 1\right)P^4(-A^2 + R_s A) + (L^2)(-P^2 + R_s P)(-A^2 + R_s A) = -c^2 R_s P^3(-A^2 + R_s A)$$

Subtract these two equations from each other:

$$c^2\left(\frac{E^2}{c^4} - 1\right)[A^4(-P^2 + R_s P) - P^4(-A^2 + R_s A)] = -c^2 R_s A^3(-P^2 + R_s P) + c^2 R_s P^3(-A^2 + R_s A)$$

$$c^2\left(\frac{E^2}{c^4} - 1\right) = \frac{-c^2 R_s A^3(-P^2 + R_s P) + c^2 R_s P^3(-A^2 + R_s A)}{[A^4(-P^2 + R_s P) - P^4(-A^2 + R_s A)]}$$

$$\left(\frac{E^2}{c^4} - 1\right) = \frac{-R_s[A^3(-P^2 + R_s P) - P^3(-A^2 + R_s A)]}{[A^4(-P^2 + R_s P) - P^4(-A^2 + R_s A)]}$$

$$\left(\frac{E^2}{c^4} - 1\right) = \frac{-R_s[A^3 P(-P + R_s) - P^3 A(-A + R_s)]}{[A^4 P(-P + R_s) - P^4 A(-A + R_s)]}$$

$$\left(\frac{E^2}{c^4} - 1\right) = \frac{-R_s AP[A^2(-P + R_s) - P^2(-A + R_s)]}{AP[A^3(-P + R_s) - P^3(-A + R_s)]}$$

$$\left(\frac{E^2}{c^4} - 1\right) = \frac{-R_s[A^2(-P + R_s) - P^2(-A + R_s)]}{[A^3(-P + R_s) - P^3(-A + R_s)]}$$

$$\left(\frac{E^2}{c^4} - 1\right) = \frac{-R_s[-PA^2 + R_s A^2 + AP^2 - R_s P^2]}{[-PA^3 + R_s A^3 + AP^3 - R_s P^3]}$$

$$\left(\frac{E^2}{c^4}-1\right)=\frac{-R_s[-AP(A-P)+R_s(A^2-P^2)]}{[-AP(A^2-P^2)+R_s(A^3-P^3)]}$$

$$\left(\frac{E^2}{c^4}-1\right)=\frac{-R_s(A-P)[-AP+R_s(A+P)]}{(A-P)\left[-AP(A+P)+R_s\frac{(A^3-P^3)}{A-P}\right]}$$

Intermezzo to work out $\frac{(A^3-P^3)}{A-P}$:

$$(A^2-P^2)(A+P)=A^3-AP^2+A^2P-P^3$$

$$A^3-P^3=(A^2-P^2)(A+P)-AP(A-P)$$

$$A^3-P^3=(A-P)(A+P)(A+P)-AP(A-P)$$

$$=>\frac{A^3-P^3}{A-P}=(A+P)^2-AP$$

Now we fill in the result:

$$\left(\frac{E^2}{c^4}-1\right)=\frac{-R_s(A-P)[-AP+R_s(A+P)]}{(A-P)[-AP(A+P)+R_s(A+P)^2-R_sAP]}$$

$$\left(\frac{E^2}{c^4}-1\right)=\frac{-R_s[-AP+R_s(A+P)]}{[-AP(A+P+R_s)+R_s(A+P)^2]}$$

$$\boldsymbol{\left(\frac{E^2}{c^4}-1\right)=\frac{R_s[-AP+R_s(A+P)]}{AP(A+P+R_s)-R_s(A+P)^2}} \tag{36a}$$

Now we can find $L^2/_{c^2}$ by applying the same method to equations (36) and (37):

$$c^2\left(\frac{E^2}{c^4}-1\right)A^4+(L^2)(-A^2+R_sA)=-c^2R_sA^3 \tag{36}$$

$$c^2\left(\frac{E^2}{c^4}-1\right)P^4+(L^2)(-P^2+R_sP)=-c^2R_sP^3 \tag{37}$$

Multiply 36 by A :

$$c^2\left(\frac{E^2}{c^4}-1\right)A^4P^4+(L^2)(-A^2+R_sA)P^4=-c^2R_sA^3P^4$$

Multiply 37 by P :

$$c^2\left(\frac{E^2}{c^4}-1\right)A^4P^4+(L^2)(-P^2+R_sP)A^4=-c^2R_sA^4P^3$$

Now subtract:

$$(L^2)[(-A^2+R_sA)P^4-(-P^2+R_sP)A^4]=-c^2R_sA^3P^4+c^2R_sA^4P^3$$

$$L^2=\frac{c^2R_sA^3P^3[-P+A]}{(-A^2+R_sA)P^4-(-P^2+R_sP)A^4}$$

$$L^2 = \frac{c^2 R_s A^3 P^3[-P+A]}{(-A+R_s)AP^4 - (-P+R_s)PA^4}$$

$$L^2 = \frac{c^2 R_s A^3 P^3[-P+A]}{AP[(-A+R_s)P^3 - (-P+R_s)A^3]}$$

$$L^2 = \frac{c^2 R_s A^2 P^2[-P+A]}{[(-A+R_s)P^3 - (-P+R_s)A^3]}$$

$$L^2 = \frac{c^2 R_s A^2 P^2[-P+A]}{A^3 P - AP^3 - (A^3 - P^3)R_s}$$

$$L^2 = \frac{c^2 R_s A^2 P^2[-P+A]}{AP(A^2 - P^2) - (A^3 - P^3)R_s}$$

$$L^2 = \frac{c^2 R_s A^2 P^2}{AP(A+P) - R_s(A+P)^2 + APR_s}$$

$$L^2 = \frac{c^2 R_s A^2 P^2}{AP(A+P+R_s) - R_s(A+P)^2}$$

$$\boldsymbol{\frac{L^2}{c^2} = \frac{R_s A^2 P^2}{AP(A+P+R_s) - R_s(A+P)^2}}$$

Finally, we get equation (36a) from above and the equation of $\frac{L^2}{c^2}$:

$$\frac{E^2}{c^4} - 1 = \frac{-APR_s + (A+P){R_s}^2}{AP(A+P+R_s) - R_s(A+P)^2}$$

$$\frac{L^2}{c^2} = \frac{A^2 P^2 R_s}{AP(A+P+R_s) - R_s(A+P)^2}$$

Next, we can introduce the variable:

$$D = \frac{AP}{A+P}$$

to further simplify the expressions. This has the dimension of distance.

Then the expression above for $E^2 - 1$ and L^2 :

$$\frac{E^2}{c^4} - 1 = \frac{(-R_s/AP) + \left({R_s}^2/DAP\right)}{\frac{1}{D} + \left(\frac{R_s}{AP}\right) - \left(\frac{R_s}{D^2}\right)}$$

$$\frac{L^2}{c^2} = \frac{R_s}{\frac{1}{D} + \left(\frac{R_s}{AP}\right) - \left(\frac{R_s}{D^2}\right)}$$

$$\frac{\frac{L^2}{c^2}}{\frac{E^2}{c^4} - 1} = \frac{R_s}{(-R_s/AP) + \left({R_s}^2/DAP\right)} = \frac{AP}{-1 + R_s/D}$$

$$\frac{\frac{L^2}{c^2 AP}}{1 - \frac{E^2}{c^4}} = \frac{1}{1 - R_s/D} \tag{38}$$

We want an expression for Ɛ, the third non-zero root of:

$$\frac{E^2/c^4-1}{L^2/c^2}r^4+\frac{R_s}{L^2/c^2}r^3-r^2+R_s r=0$$

$$\frac{E^2/c^4-1}{L^2/c^2}\left[r^4+\frac{R_s}{\frac{E^2}{c^4}-1}r^3-\frac{\frac{L^2}{c^2}}{\frac{E^2}{c^4}-1}r^2+\frac{\frac{L^2}{c^2}}{\frac{E^2}{c^4}-1}R_s r\right]=0$$

This gives the three non-zero roots: A, P, and Ɛ.
The complete expression becomes:

$$\frac{E^2/c^4-1}{L^2/c^2}(r-A)(r-P)(r-\varepsilon)r$$

Let's work out the four factors:

$$\frac{E^2/c^4-1}{L^2/c^2}[r^4-(\mathrm{A}+\mathrm{P}+\varepsilon)\mathrm{r}^3+\{\mathrm{AP}+\varepsilon(\mathrm{A}+\mathrm{P})\}\mathrm{r}^2-\varepsilon APr]$$

We know that the sum of the three non-zero roots is equal to $\frac{R_s}{E^2/c^4-1}$ (the coefficient of r^3 in the standard form of the polynomial); therefore, we obtain:

$$-(\mathrm{A}+\mathrm{P}+\varepsilon)=R_s\frac{1}{E^2/c^4-1}$$

This enables us to further analyze the relationship between the roots *A*, *P*, andε in terms of R_s , the Schwarzschild radius, and the energy and angular momentum terms.

From the above, we know that:

$$\left(\frac{E^2}{c^4}-1\right)=\frac{R_s[-AP+(A+P)R_s]}{AP(A+P+R_s)-R_s(A+P)^2}$$

So we fill this into the above equation:

$$\mathrm{A}+\mathrm{P}+\varepsilon=R_s\frac{-1}{E^2/c^4-1}$$

$$\mathrm{A}+\mathrm{P}+\varepsilon=R_s\frac{-AP(A+P+R_s)+R_s(A+P)^2}{R_s[-AP+(A+P)R_s]}=\frac{-AP(A+P+R_s)+R_s(A+P)^2}{-AP+(A+P)R_s}$$

$$\varepsilon=\frac{-AP(A+P+R_s)+R_s(A+P)^2}{-AP+(A+P)R_s}-(\mathrm{A}+\mathrm{P})$$

$$=\frac{-AP(A+P+R_s)+R_s(A+P)^2+\mathrm{AP}(A+P)-(A+P)^2R_s}{-AP+(A+P)R_s}$$

$$\varepsilon=\frac{-AP(A+P+R_s)+\mathrm{AP}(A+P)}{-AP+(A+P)R_s}=\frac{-APR_s}{-AP+(A+P)R_s}=\frac{R_s}{1-\frac{(A+P)R_s}{AP}}=\frac{R_s}{1-\frac{R_s}{D}}$$

Which gives:

$$\varepsilon=\frac{R_s}{1-\frac{R_s}{D}} \tag{39}$$

Now we can approximate (35) by writing

$$\frac{E^2/c^4-1}{L^2/c^2}r^4+\frac{R_s}{L^2/c^2}r^3-r^2+R_s r=\frac{E^2/c^4-1}{L^2/c^2}(r-A)(r-P)(r-\varepsilon)r$$

$$=\frac{1-E^2/c^4}{L^2/c^2}(A-r)(r-P)(r-\varepsilon)r.$$

We obtain:

$$\emptyset_{\boldsymbol{A}}-\emptyset_{\boldsymbol{P}}=\sqrt{\frac{L^2/c^2}{1-E^2/c^4}}\int_P^A\frac{1}{\sqrt{r(A-r)(r-P)(r-\varepsilon)}}dr$$

$$=\sqrt{\frac{L^2/c^2}{1-E^2/c^4}}\int_P^A\frac{1}{\sqrt{r^2(A-r)(r-P)\left(1-\frac{\varepsilon}{r}\right)}}dr$$

$$=\sqrt{\frac{L^2/c^2}{1-E^2/c^4}}\int_P^A\frac{1}{r\sqrt{(A-r)(r-P)}}\left(1-\frac{\varepsilon}{r}\right)^{-1/2}dr$$

Now we use the Taylor series expansion $\left(1-\frac{\varepsilon}{r}\right)^{-1/2}\approx 1+\frac{\varepsilon}{2r}$, with an error ε bounded by:

$$|\varepsilon|\leq\frac{3}{8}\left(1-\frac{\varepsilon}{r}\right)^{-5/2}\left(\frac{\varepsilon}{r}\right)^2\leq\frac{3}{8}\left(1-\frac{\varepsilon}{A}\right)^{-5/2}\left(\frac{\varepsilon}{P}\right)^2$$

which produces:

$$=\sqrt{\frac{L^2/c^2}{1-E^2/c^4}}\int_P^A\left[\frac{1}{r\sqrt{(A-r)(r-P)}}+\frac{\frac{\varepsilon}{2}}{r^2\sqrt{(A-r)(r-P)}}\right]dr \qquad (40)$$

Note:

In his article "The Precession of Mercury's Perihelion" by Owen Biesel (January 25, 2008), on page 8, the left part of the integral (40) in the numerator contains $1+\varepsilon$, but we believe, following our calculations, that it should only be *1* and have adjusted the formula accordingly.

The **first integral of (40)** (**see**4.4.6.1and4.4.6.3for details) in closed form:

$$=\int_P^A\frac{1}{r\sqrt{(A-r)(r-P)}}dr$$

$$=\frac{1}{\sqrt{AP}}\arctan\left[\frac{(A-r)(r-P)+r^2-AP}{2\sqrt{(A-r)(r-P)AP}}\right]_P^A$$

$$\rightarrow\frac{1}{\sqrt{AP}}[\arctan[+\infty]-\arctan[-\infty]]=\frac{1}{\sqrt{AP}}\left[\frac{\pi}{2}+\frac{\pi}{2}\right]=\frac{1}{\sqrt{AP}}\pi$$

The **second integral of (40)** (see4.4.6.2for details) is more difficult, but can be evaluated in closed form:

$$\int_P^A \frac{\varepsilon/2}{r^2\sqrt{(A-r)(r-P)}}dr = \frac{\pi\varepsilon/2}{2\sqrt{AP}}\,\frac{A+P}{AP} = \frac{1}{\sqrt{AP}}\,\frac{\pi\varepsilon}{4D}$$

If we now recognize that:

$$\frac{L^2/c^2AP}{1-E^2/c^4} = \frac{1}{1-R_s/D}$$

and:

$$\varepsilon = \frac{R_s}{1-R_s/D}$$

(see (38) and (39) above), then we find that:

$$\emptyset_{\boldsymbol{A}} - \emptyset_{\boldsymbol{P}} = \frac{1}{\sqrt{AP}}\pi\sqrt{\frac{L^2/c^2}{1-E^2/c^4}} + \frac{1}{\sqrt{AP}}\,\frac{\pi\varepsilon}{4D}\sqrt{\frac{L^2/c^2}{1-E^2/c^4}}$$

$$\emptyset_{\boldsymbol{A}} - \emptyset_{\boldsymbol{P}} = \pi\sqrt{\frac{L^2/c^2AP}{1-E^2/c^4}} + \frac{\pi\varepsilon}{4D}\sqrt{\frac{L^2/c^2AP}{1-E^2/c^4}} = \pi\sqrt{\frac{1}{1-R_s/D}} + \frac{\pi\varepsilon}{4D}\sqrt{\frac{1}{1-R_s/D}}$$

$$= \frac{\pi}{\sqrt{1-R_s/D}}\left(1+\frac{\varepsilon}{4D}\right) = \frac{\pi}{\sqrt{1-R_s/D}}\left(1+\frac{1}{4}\frac{1}{D}\frac{R_s}{1-R_s/D}\right)$$

$$= \frac{\pi}{\sqrt{1-R_s/D}}\left(1+\frac{1}{4}\frac{R_s/D}{1-R_s/D}\right)$$

With the observed values $A(phelion) = 69.8.10^6\text{km}$, and $P(erihelion) = 46.0.10^6\text{km}$, we obtain:

$$D = 27.7.10^6\text{km, and } R_s = \frac{2GM}{c^2} = 2.95\text{km}$$

And we can approximate the term as follows:

$$\frac{\pi}{\sqrt{1-R_s/D}}\left(1+\frac{1}{4}\frac{R_s/D}{1-R_s/D}\right) \approx \pi + 2.512\cdot 10^{-7}$$

This gives a reliable estimate of $\emptyset_{\boldsymbol{A}} - \emptyset_{\boldsymbol{P}}$ (half a revolution, in radians).
This gives:

$$\Delta\emptyset = 2.512\cdot 10^{-7}\text{ radians for halve a cycle}$$

And

$$\Delta\emptyset = 5.024\cdot 10^{-7}\text{radians for a whole cycle}$$

Mercury's orbital period is 87.969 days, so Mercury completes 415.2 revolutions per century. Since there are $360\cdot 60\cdot 60/2\pi$ arc seconds per radian, we find that Mercury's perihelion shifts by:

$$\Delta\emptyset = (5.024\cdot 10^{-7})\left(\frac{360\cdot 60\cdot 60}{2\pi}\right)\cdot 415.2 = 43.027\ \text{ arc seconds per century.}$$

$$\Delta\emptyset = \boldsymbol{43.027}\ \boldsymbol{arc\ seconds\ per\ century}.$$

Note:

According to Asaf Pe'er, for a small deflection angle, the result (see equation 6 in chapter 4.7) is:

$$\delta\phi_{prec} = \frac{6\pi GM_{sun}}{c^2 a(1-\varepsilon^2)} \tag{41}$$

Where for Mercury:

- a is the semi-major axis:$5.79 * 10^{10}$ m
- ε is the eccentricity: 0.206
- M is the mass of the sun:$1.989 * 10^{30}$ kg
- G is the gravitational constant: $6.674 * 10^{-11}\ \mathrm{Nm^2kg^{-2}}$
- c is the speed of light: $3 * 10^8\ \mathrm{m/s^2}$

$$\Delta\emptyset = \frac{6\pi GM}{a(1-e^2)c^2} = 5.02 * 10^{-7}\ \text{ rad per cycle}$$

To calculate the precession per century:

$$\Delta\emptyset = 5.02 * 10^{-7}\ * \left(100 * \frac{365.25}{88}\right) * \left(\frac{360 * 60 * 60}{2\pi}\right)$$

$$\Delta\emptyset = \mathbf{43}'' \text{ (arc seconds per century).}$$

Which leads to the same result.

This gives us the exact relationship for the precession angle of Mercury's orbit, as described in the result of 43,027 arc seconds per century.

4.4.6 Conclusion

The deviation of Mercury's orbit due to general relativity is determined by the additional curvature terms in equation (35). The actual precession per orbit can be calculated by the deviation of the integral ΔØ from 2π. This theoretical prediction corresponds to the observed deviation of approximately **43 arc seconds per century,** an effect that cannot be explained by Newtonian mechanics.

4.4.6.1 We check the first integral.

Checking the integrand:

$$\frac{d}{dr}\left\{\frac{1}{\sqrt{AP}}\arctan\left[\frac{(A-r)(r-P)+r^2-AP}{2\sqrt{(A-r)(r-P)AP}}\right]\right\} ?=? \frac{1}{r\sqrt{(A-r)(r-P)}}$$

We know that:

$$\frac{d\arctan x}{dx} = \frac{1}{1+x^2}$$

Therefore:

$$\frac{1}{\sqrt{AP}}\frac{d}{dr}\left\{\arctan\left[\frac{(A-r)(r-P)+r^2-AP}{2\sqrt{(A-r)(r-P)AP}}\right]\right\}=\frac{1}{\sqrt{AP}}\frac{1}{1+\left[\frac{(A-r)(r-P)+r^2-AP}{2\sqrt{(A-r)(r-P)AP}}\right]^2}\frac{d}{dr}\left[\frac{(A-r)(r-P)+r^2-AP}{2\sqrt{(A-r)(r-P)AP}}\right]$$

$$=\frac{1}{\sqrt{AP}}\frac{4(A-r)(r-P)AP}{4(A-r)(r-P)AP+[(A-r)(r-P)+r^2-AP]^2}\frac{d}{dr}\left[\frac{(A-r)(r-P)+r^2-AP}{2\sqrt{(A-r)(r-P)AP}}\right]$$

$$=\frac{1}{\sqrt{AP}}\frac{4(A-r)(r-P)AP}{4(A-r)(r-P)AP+[(A-r)(r-P)+r^2-AP]^2}\left[\frac{-(r-P)+(A-r)+2r}{2\sqrt{(A-r)(r-P)AP}}\right.$$
$$\left.-\frac{AP\{(A-r)(r-P)+r^2-AP\}\{-(r-P)+(A-r)\}}{4\{(A-r)(r-P)AP\}^{3/2}}\right]$$

$$=\frac{1}{\sqrt{AP}}\frac{4(A-r)(r-P)AP}{4(A-r)(r-P)AP+[(A-r)(r-P)+r^2-AP]^2}\left[\frac{A+P}{2\sqrt{(A-r)(r-P)AP}}\right.$$
$$\left.-\frac{AP\{Ar-AP-r^2+rP+r^2-AP\}\{-r+P+A-r\}}{4\{(A-r)(r-P)AP\}^{3/2}}\right]$$

$$=\frac{1}{\sqrt{AP}}\frac{4(A-r)(r-P)AP}{4(A-r)(r-P)AP+[(A-r)(r-P)+r^2-AP]^2}\left[\frac{A+P}{2\sqrt{(A-r)(r-P)AP}}-\frac{AP\{Ar-2AP+rP\}\{P+A-2r\}}{4\{(A-r)(r-P)AP\}^{3/2}}\right]$$

$$=\frac{1}{\sqrt{AP}}\frac{4(A-r)(r-P)AP}{4(A-r)(r-P)AP+[(A-r)(r-P)+r^2-AP]^2}\frac{1}{\sqrt{(A-r)(r-P)AP}}\left[\frac{2(A+P)}{4}\right.$$
$$\left.-\frac{AP\{Ar-2AP+rP\}\{P+A-2r\}}{4(A-r)(r-P)AP}\right]$$

$$=\frac{1}{\sqrt{AP}}\frac{4(A-r)(r-P)AP}{4(A-r)(r-P)AP+[(A-r)(r-P)+r^2-AP]^2}\frac{1}{\sqrt{(A-r)(r-P)AP}}*$$
$$\left[\frac{2(A+P)(A-r)(r-P)AP-AP\{Ar-2AP+rP\}\{P+A-2r\}}{4(A-r)(r-P)AP}\right]$$

$$=\frac{1}{\sqrt{AP}}\frac{2(A+P)(A-r)(r-P)AP-AP\{Ar-2AP+rP\}\{P+A-2r\}}{4(A-r)(r-P)AP+[(A-r)(r-P)+r^2-AP]^2}\frac{1}{\sqrt{(A-r)(r-P)AP}}$$

$$=\frac{1}{AP\sqrt{(A-r)(r-P)}}\frac{2(A+P)(A-r)(r-P)AP-AP\{Ar-2AP+rP\}\{P+A-2r\}}{4(A-r)(r-P)AP+[(A-r)(r-P)+r^2-AP]^2}$$

$$=\frac{1}{\sqrt{(A-r)(r-P)}}\frac{2(A+P)(A-r)(r-P)-\{Ar-2AP+rP\}\{P+A-2r\}}{4(A-r)(r-P)AP+[(A-r)(r-P)+r^2-AP]^2}$$

$$=\frac{1}{\sqrt{(A-r)(r-P)}}\frac{(2A^2-2Ar+2AP-2Pr)(r-P)-\{APr-2AP^2+rP^2+A^2r-2A^2P+APr-2Ar^2+4APr-2Pr^2\}}{4A^2Pr-4APr^2-4A^2P^2+4AP^2r+[Ar-r^2-AP+Pr+r^2-AP]^2}$$

$$=\frac{1}{\sqrt{(A-r)(r-P)}}\frac{(2A^2-2Ar+2AP-2Pr)(r-P)-\{6APr-2AP^2+P^2r+A^2r-2A^2P-2Ar^2-2Pr^2\}}{4A^2Pr-4APr^2-4A^2P^2+4AP^2r+[Ar-2AP+Pr]^2}$$

$$=\frac{1}{\sqrt{(A-r)(r-P)}}\frac{2A^2r-2Ar^2+4APr-2Pr^2-2A^2P-2AP^2+2P^2r-6APr+2AP^2-P^2r-A^2r+2A^2P+2Ar^2+2Pr^2}{4A^2Pr-4APr^2-4A^2P^2+4AP^2r+[Ar-2AP+Pr]^2}$$

$$=\frac{1}{\sqrt{(A-r)(r-P)}}\frac{A^2r-2APr++P^2r}{4A^2Pr-4APr^2-4A^2P^2+4AP^2r+[Ar-2AP+Pr]^2}$$

$$= \frac{1}{\sqrt{(A-r)(r-P)}} \frac{r(A^2-2AP++P^2)}{4A^2Pr-4APr^2-4A^2P^2+4AP^2r+[Ar-2AP+Pr]^2}$$

$$= \frac{1}{\sqrt{(A-r)(r-P)}} \frac{r(A-P)^2}{4A^2Pr-4APr^2-4A^2P^2+4AP^2r+A^2r^2+4A^2P^2+P^2r^2-4A^2Pr+2APr^2-4AP^2r}$$

$$= \frac{1}{\sqrt{(A-r)(r-P)}} \frac{r(A-P)^2}{-2APr^2+A^2r^2+P^2r^2} = \frac{1}{\sqrt{(A-r)(r-P)}} \frac{r(A-P)^2}{r^2(-2AP+A^2+P^2)}$$

This ultimately becomes:

$$= \frac{1}{\sqrt{(A-r)(r-P)}} \frac{r(A-P)^2}{r^2(A-P)^2}$$

Which results in:

$$= \frac{1}{r\sqrt{(A-r)(r-P)}}$$

So:

$$\frac{1}{r\sqrt{(A-r)(r-P)}}$$

This confirms that the integration of the integrand is correct!

4.4.6.2 Calculation of the Second Integral in the Previous Chapter.

We have derived the expression for the second integral:

General form:

$$\int \frac{1}{x^2\sqrt{ax^2+bx+c}}dx = -\frac{\sqrt{ax^2+bx+c}}{cx} - \frac{b}{2c}\int \frac{1}{x\sqrt{ax^2+bx+c}}dx$$

(See also the next chapter for the calculation of the integral on the right-hand side.)

$$\int \frac{1}{x^2\sqrt{ax^2+bx+c}}dx = -\frac{\sqrt{ax^2+bx+c}}{cx} - \frac{b}{2c\sqrt{-c}} arcsin \frac{bx+2c}{|x|\sqrt{b^2-4ac}} \quad ,(c < o)$$

Now with $a = -1, b = A + P$ and $c = -AP$

$$\emptyset_{\boldsymbol{A}} - \emptyset_{\boldsymbol{P}} = \int_P^A \frac{\varepsilon/2}{r^2\sqrt{(A-r)(r-P)}}dr = \quad \int_P^A \frac{\varepsilon/2}{r^2\sqrt{-r^2+(A+P)r-AP}}dr$$

$$= -\varepsilon/2\left[\frac{\sqrt{-r^2+(A+P)r-AP}}{-APr}\right]_P^A + \varepsilon/2\frac{(A+P)}{2AP\sqrt{AP}}\left[arcsin\frac{(A+P)r-2AP}{|r|\sqrt{(A+P)^2-4AP}}\right]_P^A$$

$$= 0 + \varepsilon/2\frac{(A+P)}{2AP\sqrt{AP}}\left\{arcsin\frac{(A+P)A-2AP}{|A|\sqrt{(A+P)^2-4AP}} - arcsin\frac{(A+P)P-2AP}{|P|\sqrt{(A+P)^2-4AP}}\right\}$$

$$= \varepsilon/2\,\frac{(A+P)}{2AP\sqrt{AP}}\left\{arcsin\frac{(A-P)A}{|A|(A-P)} - arcsin\frac{(P-A)P}{|P|(A-P)}\right\}$$

$$= \varepsilon/2\,\frac{(A+P)}{2AP\sqrt{AP}}\{\arcsin(1) - \arcsin(-1)\}$$

$$= \varepsilon/2\,\frac{(A+P)}{2AP\sqrt{AP}}\left\{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right\} = \varepsilon/2\,\frac{\pi(A+P)}{2AP\sqrt{AP}} = \frac{\pi\varepsilon}{4D\sqrt{AP}}$$

This corresponds with the calculations.

4.4.6.3 Alternative Solution for Integral 1.

According to the solutions given in Wikipedia: https://nl.wikipedia.org/wiki/Lijst_van_integralen, we have:

$$\int \frac{1}{x\sqrt{ax^2+bx+c}}dx = \frac{1}{\sqrt{-c}}arcsin\frac{bx+2c}{|x|\sqrt{b^2-4ac}} + C, (c<0)$$

So:

$$\emptyset_{\boldsymbol{A}} - \emptyset_{\boldsymbol{P}} = \int_P^A \frac{1}{r\sqrt{(A-r)(r-P)}}dr = \int_P^A \frac{1}{r\sqrt{-r^2+(A+P)r-AP}}dr$$

$$= \frac{1}{\sqrt{AP}}\arcsin\left[\frac{(A+P)r-2AP}{|r|\sqrt{(A+P)^2-4AP}}\right]_P^A$$

$$= \frac{1}{\sqrt{AP}}\left\{\arcsin\frac{(A+P)A-2AP}{|A|\sqrt{(A+P)^2-4AP}} - \arcsin\frac{(A+P)P-2AP}{|P|\sqrt{(A+P)^2-4AP}}\right\}$$

$$= \frac{1}{\sqrt{AP}}\left\{\arcsin\frac{(A-P)A}{|A|(A-P)} - \arcsin\frac{(P-A)P}{|P|(A-P)}\right\}$$

$$= \frac{1}{\sqrt{AP}}\{\arcsin(1) - \arcsin(-1)\}$$

$$= \frac{1}{\sqrt{AP}}\left\{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right\} = \frac{\pi}{\sqrt{AP}}$$

4.4.6.4 Detailed Calculation of the Time T of a Revolution.

$$L = r^2\frac{d\emptyset}{d\tau} \Rightarrow d\tau = \frac{r^2}{L}d\emptyset \Rightarrow T = \int d\tau = \int_{\mathbf{0}}^{\mathbf{2\pi}} \frac{r^2}{L}d\emptyset$$

Using equation 40:

$$d\emptyset = \sqrt{\frac{L^2/c^2}{1-E^2/c^4}}\left[\frac{1}{r\sqrt{(A-r)(r-P)}} + \frac{\frac{\varepsilon}{2}}{r^2\sqrt{(A-r)(r-P)}}\right]dr \qquad (40)$$

$$d\tau = \frac{r^2}{L}\sqrt{\frac{L^2/c^2}{1-E^2/c^4}}\left[\frac{1}{r\sqrt{(A-r)(r-P)}} + \frac{\frac{\varepsilon}{2}}{r^2\sqrt{(A-r)(r-P)}}\right]dr$$

$$d\tau = \frac{1}{L}\sqrt{\frac{L^2/c^2}{1-E^2/c^4}}\left[\frac{r}{\sqrt{(A-r)(r-P)}} + \frac{\frac{\varepsilon}{2}}{\sqrt{(A-r)(r-P)}}\right]dr$$

$$\Delta T = \int d\tau = \frac{2}{L}\sqrt{\frac{L^2/c^2}{1-E^2/c^4}}\int_P^A\left[\frac{r}{\sqrt{(A-r)(r-P)}} + \frac{\frac{\varepsilon}{2}}{\sqrt{(A-r)(r-P)}}\right]dr$$

First, the evaluation of the left integral:

$$\int_P^A \frac{r}{\sqrt{(A-r)(r-P)}}dr = \int_P^A \frac{r}{\sqrt{-r^2+(A+P)r-AP}}dr \qquad (41)$$

According to the list of integrals (Wikipedia): (https://nl.wikipedia.org/wiki/Lijst_van_integralen)

$$\int \frac{x}{\sqrt{ax^2+bx+c}}dx = \frac{\sqrt{ax^2+bx+c}}{a} - \frac{b}{2a}\int \frac{1}{\sqrt{ax^2+bx+c}}dx \qquad (42)$$

And:

$$\int \frac{1}{\sqrt{ax^2+bx+c}}dx = \frac{1}{\sqrt{-a}}\arcsin\frac{-2ax-b}{\sqrt{b^2-4ac}} + C, (a<0)$$

To convert the left integral to the integral formula:

$$\int_P^A \frac{r}{\sqrt{-r^2+(A+P)r-AP}}dr =$$

$$= \left[\frac{\sqrt{-r^2+(A+P)r-AP}}{-1}\right]_P^A - \frac{(A+P)}{-2}\int_P^A \frac{1}{\sqrt{-r^2+(A+P)r-AP}}dr$$

$$= -\sqrt{-A^2+(A+P)A-AP} + \sqrt{-P^2+(A+P)P-AP} + \frac{(A+P)}{2}\int_P^A \frac{1}{\sqrt{-r^2+(A+P)r-AP}}dr$$

$$= -0 + 0 + \frac{(A+P)}{2}\int_P^A \frac{1}{\sqrt{-r^2+(A+P)r-AP}}dr$$

Now only the integral:

$$\int_P^A \frac{1}{\sqrt{-r^2+(A+P)r-AP}}dr = \left[\arcsin\frac{2r-(A+P)}{\sqrt{(A+P)^2-4AP}} + C\right]_P^A =$$

$$\arcsin\frac{2A-(A+P)}{\sqrt{(A+P)^2-4AP}} + C - \arcsin\frac{2P-(A+P)}{\sqrt{(A+P)^2-4AP}} - C$$

$$= \arcsin\frac{A-P}{A-P} - \arcsin\frac{-A+P}{A-P}$$
$$\frac{\pi}{2} + \frac{\pi}{2} = \pi$$

So, the left integral yields:

$$\frac{(A+P)\pi}{2}$$

The right integral yields:

$$\pi\frac{\varepsilon}{2}$$

The sum is:

$$\frac{\pi}{2}\left((A+P)+\varepsilon\right)$$

So, the total integral for a complete revolution is:

$$\Delta T = 2\frac{1}{L}\sqrt{\frac{L^2/c^2}{1-E^2/c^4}}\frac{\pi}{2}\left((A+P)+\varepsilon\right)$$

With:

$$\varepsilon = \frac{R_s}{1-\frac{R_s}{D}}$$

$$\Delta T = 2\frac{1}{L}\sqrt{\frac{L^2/c^2}{1-E^2/c^4}}\frac{\pi}{2}\left((A+P)+\frac{R_s}{1-\frac{R_s}{D}}\right)$$

$$\Delta T = 2\pi\frac{A+P}{2L}\sqrt{\frac{L^2/c^2}{1-E^2/c^4}}\left(1+\frac{R_s}{(A+P)\left(1-\frac{R_s}{D}\right)}\right)$$

$$\Delta T = 2\pi\frac{A+P}{2L}\sqrt{\frac{AP}{1-R_s/D}}\left(1+\frac{R_s}{(A+P)\left(1-\frac{R_s}{D}\right)}\right)$$

For Mercury:

$$A = 6.98 * 10^{10}, P = 4.60 * 10^{10}, D = 2.77 * 10^{10}, R_{s(sun)} = 2953.25, L = 2.71 * 10^{15},$$

The time for one revolution is:

$$\Delta\boldsymbol{T} = 7598744\ sec => \frac{7598744}{24 * 3600} = \mathbf{87.95\ days}$$

Derived in chapter Schwarzschild Approach4.8.2**Error! Reference source not found.** equation $2d$ the instantaneous rotational velocity of Mercury as a function of Ø:

$$v = \left\{ \frac{GM_{sun}}{a(1-e^2)} \left(1 + 2e\cos[\emptyset(1-\epsilon)] + e^2\right) \right\}^{\frac{1}{2}} \tag{42a}$$

4.4.7 Physical meaning

- **Why precession?**
  Due to the curvature of space-time around the sun, Mercury's orbit is not a perfect ellipse, but an ellipse that slowly rotates.
- **No Newtonian explanation:**
  This effect cannot be explained by classical mechanics or the influence of other planets alone.
- **Empirical confirmation:**
  The measured value of 43 arc seconds per century was one of the first major successes of general relativity.

4.4.8 Key insight

- The precession of Mercury's perihelion is a direct, measurable consequence of the curvature of space-time as predicted by general relativity and the Schwarzschild metric.
- The quantitative agreement between theory and observation is one of the most powerful confirmations of Einstein's theory.

4.5 Experiment 5 - Shapiro Time Delay

Introduction and Physical Concept

Shapiro time delay is the effect whereby a light signal (or radar wave) traveling past a massive object (such as the sun) takes longer than expected based on a straight line in flat space-time. This is a direct consequence of the curvature of space-time by mass, as predicted by general relativity.

History:

The effect was predicted in 1964 by Irwin Shapiro and has since been confirmed in many experiments, including sending radar signals to Venus and Mercury and measuring the return time.

In the Shapiro experiment, radar signals were sent from Earth to a planet that was on the other side of the Sun at the time. These signals bounced back to Earth. According to the general theory of relativity, the signal, which just grazes the sun, will be deflected by the sun's gravity, or rather, the sun's mass has distorted space-time in such a way that the signal follows a "straight curved" line (see Fig. 4).

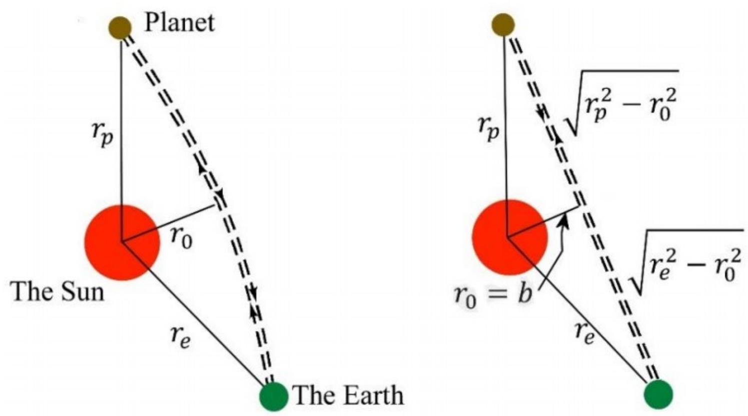


Figure 1: The radar reflection of photons from Earth to a planet and back. The left image shows the actual path, exaggerated for clarity. The right image shows the Euclidean shape.

(From *Tests of General Relativity: A Review* by Estelle Asmodelle (Asmodelle, 2017))

To define the Shapiro delay, we assume that the Earth and the planet are stationary, while the total time for the return trip of the radar signal is Δt , in coordinate time. The value of *t* must be expressed in terms of *r* over the entire path, where r_0 is the shortest distance to the Sun.

4.5.1 Derivation based on the Schwarzschild metric

The Schwarzschild equation is used to calculate the Shapiro delay.

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta \, d\emptyset^2$$

Where:

$$\sigma = \sqrt{1 - \frac{2GM_{sun}}{\boldsymbol{c^2} r}} = \sqrt{1 - \frac{R_s}{r}}$$

and

$$R_s = \frac{2GM_{sun}}{\boldsymbol{c^2}}$$

The Schwarzschild radius of the Sun.

We choose the frame of reference such that it corresponds to the equatorial plane $(\theta = \pi/2)$.
Then the following applies:

$$c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\emptyset^2$$

For photons or radar echoes, $d\tau = 0$. In that case, the following applies:

$$\sigma^2 c^2 dt^2 = \frac{dr^2}{\sigma^2} + r^2 d\emptyset^2$$

Derivation to the affine parameter λ :

$$\sigma^2 c^2 \left(\frac{dt}{d\lambda}\right)^2 = \frac{1}{\sigma^2}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\emptyset}{d\lambda}\right)^2$$

As derived in formula 11 from section4.2the angular momentum is:

$$L = r^2 \frac{d\emptyset}{d\lambda}$$

$$\sigma^2 c^2 \left(\frac{dt}{d\lambda}\right)^2 = \frac{1}{\sigma^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2}$$

Multiply by σ^2:

$$\sigma^4 c^2 \left(\frac{dt}{d\lambda}\right)^2 = \left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2}\sigma^2$$

Suppose:

$$k^2 = \sigma^4 \left(\frac{dt}{d\lambda}\right)^2$$

Note**:** This is also $k = \frac{E}{c^2}$ as can be seen in formula 9a in section4.2.
Then:

$$\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2}\sigma^2 = k^2 c^2$$

The energy equation for a photon orbit in Schwarzschild geometry is:

$$\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2}\left(1 - \frac{R_s}{r}\right) = k^2 c^2 \tag{42b}$$

As previously derived:

$$k^2 = \sigma^4 \left(\frac{dt}{d\lambda}\right)^2 \;\Rightarrow\; \left(\frac{dt}{d\lambda}\right)^2 = \frac{k^2}{\sigma^4}$$

Where we use:

$$\left(\frac{dt}{d\lambda}\right)^2 = \frac{k^2}{\sigma^4} = \frac{k^2}{\left(1 - \frac{R_s}{r}\right)^2}$$

Now:

$$\left(\frac{dr}{d\lambda}\right)^2 = \left(\frac{dr}{dt}\frac{dt}{d\lambda}\right)^2 = \frac{k^2}{\left(1 - \frac{R_s}{r}\right)^2}\left(\frac{dr}{dt}\right)^2 \tag{42c}$$

We can rewrite the energy equation (42b):

$$\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2}\left(1 - \frac{R_s}{r}\right) = k^2 c^2$$

Replace $\left(\frac{dr}{d\lambda}\right)^2$ with (42c):

$$\frac{k^2}{\left(1-\frac{R_s}{r}\right)^2}\left(\frac{dr}{dt}\right)^2+\frac{L^2}{r^2}\left(1-\frac{R_s}{r}\right)=k^2c^2$$

Divide by $\left(1-\frac{R_s}{r}\right)$:

$$\frac{k^2}{\left(1-\frac{R_s}{r}\right)^3}\left(\frac{dr}{dt}\right)^2+\frac{L^2}{r^2}-\frac{k^2c^2}{\left(1-\frac{R_s}{r}\right)}=0$$

Then we divide by k^2 :

$$\frac{1}{\left(1-\frac{R_s}{r}\right)^3}\left(\frac{dr}{dt}\right)^2+\frac{L^2}{k^2r^2}-\frac{c^2}{1-\frac{R_s}{r}}=0 \tag{43}$$

Now consider the path of a photon from Earth to another planet (for example, Venus, with $r_p = r_V$), as shown in Figure 2. It is clear that the path of the photon will be deflected by the gravitational field of the Sun. Let r_0 be the coordinate distance of the closest point where the photon approaches the Sun; then:

$$\left(\frac{dr}{dt}\right)_{r_0}=0$$

Then we find from (43) the relationship between the constants:

$$\frac{L^2}{k^2{r_0}^2}=\frac{c^2}{1-\frac{R_s}{r_0}}$$

After rearranging, we can write (43) as:

$$\left(\frac{dr}{dt}\right)^2=\left(1-\frac{R_s}{r}\right)^3\left(-\frac{L^2}{k^2r^2}+\frac{c^2}{1-\frac{R_s}{r}}\right)=\left(1-\frac{R_s}{r}\right)^3\left(\frac{c^2}{1-\frac{R_s}{r}}-\frac{L^2{r_0}^2}{k^2{r_0}^2r^2}\right)$$

$$=\left(1-\frac{R_s}{r}\right)^3\left(\frac{c^2}{1-\frac{R_s}{r}}-\frac{{r_0}^2c^2}{r^2\left(1-\frac{R_s}{r_0}\right)}\right)$$

$$=\left(1-\frac{R_s}{r}\right)^2\left(c^2-\frac{{r_0}^2c^2\left(1-\frac{R_s}{r}\right)}{r^2\left(1-\frac{R_s}{r_0}\right)}\right)=c^2\left(1-\frac{R_s}{r}\right)^2\left(1-\frac{{r_0}^2\left(1-\frac{R_s}{r}\right)}{r^2\left(1-\frac{R_s}{r_0}\right)}\right)$$

$$\Rightarrow\ \frac{dr}{dt}=c\left(1-\frac{R_s}{r}\right)\left[1-\frac{{r_0}^2\left(1-\frac{R_s}{r}\right)}{r^2\left(1-\frac{R_s}{r_0}\right)}\right]^{\frac{1}{2}}$$

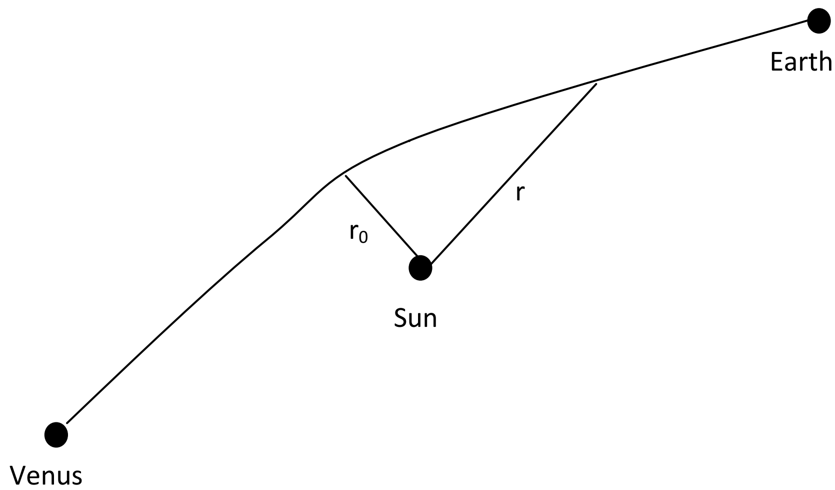


Figure 2 Photon path from Earth to Venus, deflected by the Sun.

This can be integrated to determine the time required to travel between points r_0 and *r:*

$$t(r, r_0) = \int_{r_0}^{r} \frac{1}{c\left(1 - \frac{R_s}{r}\right)\left[1 - \frac{{r_0}^2\left(1 - \frac{R_s}{r}\right)}{r^2\left(1 - \frac{R_s}{r_0}\right)}\right]^{\frac{1}{2}}} dr$$

4.5.1.1 First-order approximation

Since $R_s \ll r_0$, we can take the first-order Taylor approximation of:

$$\frac{\left(1 - \frac{R_s}{r}\right)}{\left(1 - \frac{R_s}{r_0}\right)} \approx \left(1 - \frac{R_s}{r}\right)\left(1 + \frac{R_s}{r_0}\right) = 1 - \frac{R_s}{r} + \frac{R_s}{r_0} - \frac{{R_s}^2}{r r_0}$$

So the integrand can be expanded to the first order in R_s/r :

$$t(r, r_0) = \int_{r_0}^{r} \frac{1}{c\left(1 - \frac{R_s}{r}\right)\left[1 - \frac{{r_0}^2}{r^2}\left(1 - \frac{R_s}{r} + \frac{R_s}{r_0} - \frac{{R_s}^2}{r r_0}\right)\right]^{\frac{1}{2}}} dr$$

Multiply the numerator and denominator by *r:*

$$t(r, r_0) = \int_{r_0}^{r} \frac{r}{c\left(1 - \frac{R_s}{r}\right)\left[r^2 - {r_0}^2\left(1 - \frac{R_s}{r} + \frac{R_s}{r_0} - \frac{{R_s}^2}{r r_0}\right)\right]^{\frac{1}{2}}} dr$$

$$t(r, r_0) = \int_{r_0}^{r} \frac{r}{c\left(1 - \frac{R_s}{r}\right)\left[r^2 - {r_0}^2 - R_s r_0 + \frac{R_s {r_0}^2}{r} + \frac{{R_s}^2 r_0}{r}\right]^{\frac{1}{2}}} dr$$

$$t(r,r_0)=\int_{r_0}^{r}\frac{r}{c\sqrt{r^2-{r_0}^2}\left(1-\frac{R_s}{r}\right)\left[1-\frac{R_s r_0(1-\frac{r_0}{r}-\frac{R_s}{r}}{r^2-{r_0}^2}\right]^{\frac{1}{2}}}dr$$

$$t(r,r_0)=\int_{r_0}^{r}\frac{r}{c\sqrt{r^2-{r_0}^2}\left[\left(1-\frac{2R_s}{r}+\frac{{R_s}^2}{r^2}\right)\left(1-\frac{R_s r_0(1-\frac{r_0}{r}-\frac{R_s}{r}}{r^2-{r_0}^2}\right)\right]^{\frac{1}{2}}}dr$$

First, we work out the right-hand side of the numerator:

$$\left(1-\frac{2R_s}{r}+\frac{{R_s}^2}{r^2}\right)\left(1-\frac{R_s r_0(1-\frac{r_0}{r}-\frac{R_s}{r}}{r^2-{r_0}^2}\right)=$$

$$=\mathbf{1}-\frac{\mathbf{2R_s}}{\boldsymbol{r}}+\frac{{R_s}^2}{r^2}-\frac{\boldsymbol{R_s r_0}\left(\mathbf{1}-\frac{\boldsymbol{r_0}}{\boldsymbol{r}}-\frac{\boldsymbol{R_s}}{\boldsymbol{r}}\right)}{\boldsymbol{r^2}-\boldsymbol{r_0}^{\mathbf{2}}}+\frac{2{R_s}^2 r_0(1-\frac{r_0}{r}-\frac{R_s}{r}}{r(r^2-{r_0}^2)}-\frac{{R_s}^3 r_0\left(1-\frac{r_0}{r}-\frac{R_s}{r}\right)}{r^2(r^2-{r_0}^2)}$$

After ignoring the smallest terms:

$$\left(1-\frac{2R_s}{r}+\frac{{R_s}^2}{r^2}\right)\left(1-\frac{R_s r_0(1-\frac{r_0}{r}-\frac{R_s}{r}}{r^2-{r_0}^2}\right)=\mathbf{1}-\frac{\mathbf{2R_s}}{\boldsymbol{r}}-\frac{\boldsymbol{R_s r_0}\left(\mathbf{1}-\frac{\boldsymbol{r_0}}{\boldsymbol{r}}\right)}{\boldsymbol{r^2}-\boldsymbol{r_0}^{\mathbf{2}}}$$

$$\left(1-\frac{2R_s}{r}+\frac{{R_s}^2}{r^2}\right)\left(1-\frac{R_s r_0(1-\frac{r_0}{r}-\frac{R_s}{r}}{r^2-{r_0}^2}\right)=\mathbf{1}-\frac{\mathbf{2R_s}}{\boldsymbol{r}}-\frac{\boldsymbol{R_s r_0(r-r_0)}}{\boldsymbol{r(r+r_0)(r-r_0)}}$$

$$\left(1-\frac{2R_s}{r}+\frac{{R_s}^2}{r^2}\right)\left(1-\frac{R_s r_0(1-\frac{r_0}{r}-\frac{R_s}{r}}{r^2-{r_0}^2}\right)=\mathbf{1}-\frac{\mathbf{2R_s}}{\boldsymbol{r}}-\frac{\boldsymbol{R_s r_0}}{\boldsymbol{r(r+r_0)}}$$

Fill in the denominator:

$$t(r,r_0)=\int_{r_0}^{r}\frac{r}{c\sqrt{r^2-{r_0}^2}\left[\mathbf{1}-\frac{\mathbf{2R_s}}{\boldsymbol{r}}-\frac{\boldsymbol{R_s r_0}}{\boldsymbol{r(r+r_0)}}\right]^{\frac{1}{2}}}dr$$

With another first-order Taylor approximation, we obtain:

$$t(r,r_0)=\int_{r_0}^{r}\frac{r}{c\sqrt{r^2-{r_0}^2}}\left[1+\frac{R_s}{r}+\frac{R_s r_0}{2r(r+r_0)}\right]dr$$

This can be reduced to (see check below):

$$t(r,r_0)=\frac{(r^2-{r_0}^2)^{\frac{1}{2}}}{c}+\frac{R_s}{c}ln\left[\frac{r+(r^2-{r_0}^2)^{\frac{1}{2}}}{r_0}\right]+\frac{R_s}{2c}\left(\frac{r-r_0}{r+r_0}\right)^{\frac{1}{2}}$$

We can verify the above formula by taking the derivative; it must be equal to the integrand:

$$\frac{dt(r,r_0)}{dr}=\frac{r}{c(r^2-{r_0}^2)^{\frac{1}{2}}}+\frac{R_s}{c}\frac{\left(\frac{1}{r_0}+\frac{r}{r_0(r^2-{r_0}^2)^{\frac{1}{2}}}\right)}{\frac{r+(r^2-{r_0}^2)^{\frac{1}{2}}}{r_0}}+\frac{R_s}{4c}\frac{\left(\frac{1}{r+r_0}-\frac{(r-r_0)}{(r+r_0)^2}\right)}{\left(\frac{r-r_0}{r+r_0}\right)^{\frac{1}{2}}}$$

$$\frac{dt(r,r_0)}{dr}=\frac{r}{c(r^2-{r_0}^2)^{\frac{1}{2}}}+\frac{R_s}{c}\frac{\left(1+\frac{r}{(r^2-{r_0}^2)^{\frac{1}{2}}}\right)}{r+(r^2-{r_0}^2)^{\frac{1}{2}}}+\frac{R_s}{4c}\frac{\left(\frac{r+r_0-r+r_0}{(r+r_0)^2}\right)}{\left(\frac{r-r_0}{r+r_0}\right)^{\frac{1}{2}}}$$

$$\frac{dt(r,r_0)}{dr}=\frac{r}{c(r^2-{r_0}^2)^{\frac{1}{2}}}+\frac{R_s}{c}\frac{\left(r+(r^2-{r_0}^2)^{\frac{1}{2}}\right)}{\left(r+(r^2-{r_0}^2)^{\frac{1}{2}}\right)(r^2-{r_0}^2)^{\frac{1}{2}}}+\frac{R_s}{4c}\frac{(r+r_0-r+r_0)}{\left(\frac{r-r_0}{r+r_0}\right)^{\frac{1}{2}}(r+r_0)^2}$$

$$\frac{dt(r,r_0)}{dr}=\frac{r}{c(r^2-{r_0}^2)^{\frac{1}{2}}}+\frac{R_s}{c}\frac{1}{(r^2-{r_0}^2)^{\frac{1}{2}}}+\frac{R_s}{2c}\frac{r_0}{\frac{(r^2-{r_0}^2)^{\frac{1}{2}}}{r+r_0}(r+r_0)^2}$$

$$\frac{dt(r,r_0)}{dr}=\frac{r}{c(r^2-{r_0}^2)^{\frac{1}{2}}}+\frac{R_s}{c}\frac{1}{(r^2-{r_0}^2)^{\frac{1}{2}}}+\frac{R_s}{2c}\frac{r_0}{(r^2-{r_0}^2)^{\frac{1}{2}}(r+r_0)}$$

$$\frac{dt(r,r_0)}{dr}=\frac{r}{c(r^2-{r_0}^2)^{\frac{1}{2}}}\left[1+\frac{R_s}{r}+\frac{R_sr_0}{2r(r+r_0)}\right]$$

So the formula is correct!

Therefore:

$$t(r,r_0)=\frac{(r^2-{r_0}^2)^{\frac{1}{2}}}{c}+\frac{R_s}{c}ln\left[\frac{r+(r^2-{r_0}^2)^{\frac{1}{2}}}{r_0}\right]+\frac{R_s}{2c}\left(\frac{r-r_0}{r+r_0}\right)^{\frac{1}{2}}$$

The first term on the right-hand side is exactly what we would expect if light traveled in a straight line. The second and third terms give the extra coordination time needed for the photon to travel along the *curved path* to point *r*. As can be seen in Figure 2, if we send a radar beam to Venus and back, the extra coordination time relative to a straight line is:

$$\Delta t=2\left[t(r_E,r_0)+t(r_V,r_0)-\frac{({r_E}^2-{r_0}^2)^{\frac{1}{2}}}{c}-\frac{({r_V}^2-{r_0}^2)^{\frac{1}{2}}}{c}\right]$$

As mentioned earlier, the first two terms inside these parentheses form the relativistic time from Earth to Venus, and the two terms on the right form the time if the path were simply a straight line. The factor of 2 is included because the photon must go to Venus and back to Earth.

Since $r_{E\gg}r_0$ and $r_{v\gg}r_0$, we have:

$$t(r_E, r_0) - \frac{(r_E{}^2 - r_0{}^2)^{\frac{1}{2}}}{c} \approx \frac{R_s}{c} ln\left(\frac{r_E + r_E}{r_0}\right) + \frac{R_s}{2c} = \frac{R_s}{c} ln\left(\frac{2r_E}{r_0}\right) + \frac{R_s}{2c}$$

$$t(r_V, r_0) - \frac{(r_V{}^2 - r_0{}^2)^{\frac{1}{2}}}{c} \approx \frac{R_s}{c} ln\left(\frac{r_V + r_V}{r_0}\right) + \frac{R_s}{2c} = \frac{R_s}{c} ln\left(\frac{2r_V}{r_0}\right) + \frac{R_s}{2c}$$

Summation:

$$\frac{R_s}{c} ln\left(\frac{2r_E}{r_0}\right) + \frac{R_s}{c} ln\left(\frac{2r_V}{r_0}\right) + \frac{R_s}{c} = \frac{2GM}{c^3}\left[ln\left(\frac{4r_E r_V}{r_0{}^2}\right) + 1\right]$$

So to go to Venus and back, the extra coordinate time delay is:

$$\boxed{\boldsymbol{\Delta t \approx \frac{4GM_{sun}}{c^3}\left[ln\left(\frac{4r_E r_V}{r_0{}^2}\right) + 1\right]}}$$

This also shows that the time delay increases as the impact parameter r_0 (the distance to the center of gravity) decreases.

Numerical Values

- For Venus, when it is opposite Earth on the other side of the Sun:

$$\Delta t \approx 252\mu s.$$

- While for Mercury:

$$\Delta t \approx 240\mu s.$$

- Distance Venus-Sun(r_V) : $108 * 10^9\ m$
- Distance Sun-Earth(r_E) : $150 * 10^9\ m$
- Total distance Venus-Earth: $258 * 10^9\ m$

The total travel time (Earth, Sun, Venus, and back) without delay is 1720 seconds. The Shapiro delay is therefore a small but measurable effect.

4.5.1.2 Proper time of Earth versus coordinate time

Of course, clocks on Earth do not measure coordinate time, due to the rotation of the Earth on its own axis and the effect of the Earth's rotation around the Sun.
Due to the Earth's rotation around its own axis, the corresponding proper time of the signal is given by:

$$\Delta\tau = \left(1 - \frac{2GM_E}{c^2 r_E}\right)^{\frac{1}{2}} \Delta t$$

The effect is therefore:

$$\Delta t - \Delta\tau = \Delta t - \left(1 - \frac{2GM_E}{c^2 r_E}\right)^{\frac{1}{2}} \Delta t$$

This gives:

$$=> 6.98 * 10^{-10} \Delta t \quad \text{for } 252\mu s => 1.76 * 10^{-13} seconds = 0.176ps$$

$$p = 10^{-12}$$

Since $r_E \gg \frac{GM}{c^2}$, and therefore $0.176ps \ll 252\mu s$, we can ignore this effect for the accuracy of our calculation.

The effect of the Earth's rotation around the Sun causes a delay of 15 nanoseconds per second, as mentioned in chapter (4.6).
For the additional time delay $\Delta t \approx 252\ \mu s$ from Venus, the rotation of the Earth around the Sun causes a small effect of: $252 * 10^{-6} * 15 * 10^{-9} = 3.78 * 10^{-12}\ seconds = 3.78\ ps$, which can also be ignored.

4.5.2 Physical Interpretation

- The extra time delay is a direct consequence of the curvature of space-time by the Sun.
- The effect is greatest when the signal passes close to the Sun (small r_0).
- Experiments show that the measured time delay corresponds exactly to the predictions of general relativity.

4.5.3 Practical importance

- The Shapiro time delay is important for accurate navigation of space missions and testing alternative theories of gravity.
- The effect is also used in pulsar timing and in interpreting signals from spacecraft.

4.5.4 Key insight

The Shapiro time delay is one of four classic experiments that confirm general relativity. The effect is small but measurable and can be fully explained by the Schwarzschild metric.

4.6 Time relationship between observer on Earth and the center of the Sun

When we consider the deflection of light or the orbits of planets around the Sun, a frame of reference is used with the center at the center of the Sun, while we observe the phenomenon from Earth and **have** a **rotational speed relative to the Sun**. In this chapter, we investigate the time relationship between an observer on Earth and the center of the Sun, with the associated correction factors.

The starting point is the Schwarzschild metric, which describes the space-time around a spherically symmetric massive object. The metric is given by:

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta \, d\emptyset^2$$

where:

$$\sigma = \sqrt{1 - \frac{2GM_{sun}}{c^2 r}}, \qquad R_s = \frac{2GM_{sun}}{c^2}$$

- G is the gravitational constant,
- M_{sun} is the mass of the Sun,
- c is the speed of light,
- R is the distance to the center of the Sun.

The coordinates θ and $\emptyset$ represent the usual spherical coordinates, with the observations located in the equatorial plane of the Sun, so that $\theta = \pi/2$ and the radius r is constant.

4.6.1 Simplification of the Schwarzschild metric

In the case of an observer on Earth, it is assumed that the Earth is in a circular orbit around the Sun. The time measurement of the observer on Earth is associated with the proper time $d\tau$, which is the time measured by the observer himself, and the coordinate time *dt,* which corresponds to the time in the universal reference frame of the Sun.

The Schwarzschild metric is simplified to:

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - r^2 d\emptyset^2$$

where τ is the proper time of the observer on Earth and *t is* the coordinate time in the reference frame of the Sun. This can be rewritten as:

$$d\tau^2 = \sigma^2 dt^2 - \frac{r^2}{c^2}\left(\frac{d\emptyset}{dt}\right)^2 dt^2 = \left(1 - \frac{R_s}{r} - \frac{r^2}{c^2}\left(\frac{d\emptyset}{dt}\right)^2\right) dt^2$$

4.6.2 Time delay due to gravity and Earth's motion

For an observer on Earth, the time relationship is given as follows:

$$d\tau^2 = \left(1 - \frac{R_s}{r} - \frac{v^2}{c^2}\right) dt^2$$

$$d\tau = \sqrt{\left(1 - \frac{R_s}{r} - \frac{v^2}{c^2}\right)} dt$$

where R_s is the Schwarzschild radius of the Sun, *v is* the speed of the Earth in its orbit, and *r is* the distance from the Earth to the center of the Sun. This is the general time relationship that takes into account both the gravity of the Sun and the speed of the Earth.

The specific values are as follows:

- R_s=2950 m,
- $v = 30.000 \text{ m/s}$,
- $r = 150\ x\ 10^9 \text{ m}$ (the average distance from the Earth to the Sun).

By filling in the values and expanding the expression for$d\tau$ with a Taylor series to the first order, we obtain the following approximation:

$$d\tau = \left(1 - \frac{R_s}{2r} - \frac{v^2}{2c^2}\right)dt$$

The second term on the right-hand side is the result of the Sun's gravitational pull, and the third term is the result of the Earth's speed around the Sun.
Substituting the numerical values gives:

$$d\tau = (1 - 99.10^{-10} - 50.10^{-10})dt$$

$$\boldsymbol{d\tau \approx \left(1 - 15.10^{-9}\right)dt}$$

$$\Delta t - \Delta\tau = 15.10^{-9}\Delta t$$

This means that the observer's proper time on Earth (during its orbit around the Sun) is delayed compared to universal time, with a delay of approximately 15.10^{-9} of the transit time *dt*. So over one second, the difference is 15 nanoseconds.

This is the relationship between the time of the observer on Earth and the universal Sun reference time *t*.

4.6.3 Correction factor for Earth's gravity

The observer on Earth is also influenced by Earth's gravity. This gravity must also be taken into account for a complete description of the time relationship. In this case, the proper time $d\tau$ is adjusted by Earth's gravity, using the following metric for Earth:

$$d\tau = \sqrt{1 - \frac{2GM_E}{c^2 r_e}}dt$$

where M_E is the mass of the Earth, *r* is the radius of the Earth, and the values of the constants *G* and *c* are known. The mass of the Earth is $M_E = 5.9742\ x\ 10^{24}\text{kg}$, and the radius of the Earth is $r_e = 6.381\ x\ 10^6\text{m}$.

This gives:

$$d\tau = \sqrt{1 - 1.3908 * 10^{-9}}dt = (1 - 0.6954 * 10^{-9})dt$$

For an observer at the equator, the rotational speed v_{rot} of the Earth is also important. The angular velocity $\frac{d\emptyset}{dt}$ of the Earth is given by the rotation period of the Earth (sidereal period: 86162.4 seconds):

$$\frac{d\emptyset}{dt} = \frac{2\pi}{T_{tot}} = 7.2923\ x\ 10^{-5}\text{rad/s}$$

The adjusted time relationship for an observer on the equator, including the Earth's rotation, then becomes:

$$d\tau = \sqrt{\left(1 - \frac{R_E}{r_e} - \frac{{v_E}^2}{c^2}\right)}dt$$

where the second term concerns the contribution of the Earth's rotational speed. By filling in the correct values, we obtain the final time relation:

$$d\tau = \sqrt{1 - 1.3908 * 10^{-9} - 2.4059 * 10^{-12}}dt = (1 - 0.6966 * 10^{-9})dt.$$

Where:

- $R_E = 0.008875\ \text{m}$ (Schwarzschild radius of the Earth)
- r_e= 6,381,000 m (radius of the Earth)
- $v_E = 465\ \text{m/s}$ (rotation of the Earth around its own axis)
- $c = 3 \cdot 10^8$ m/s

4.6.4 Conclusion

The time relationship between the observer on Earth and the center of the Sun is a combination of effects arising from the Sun's gravity, the speed of Earth in its orbit, and the gravity of Earth itself. These factors result in a time delay that initially appears to be caused by the interaction of the Earth with the Sun's gravitational field, but is also influenced by the Earth's motion and the Earth's local gravity.

4.6.5 Physical significance

- Clocks on Earth run slower than a hypothetical clock at the center of the Sun, due to both gravity and motion.
- For GPS and space travel, these corrections are essential for accurate timekeeping.

4.7 Alternative Derivation of the Orbital Equation

According to Kepler's first law, all planetary orbits around the Sun are elliptical. As we saw in **section4.4**General Relativity has shown that there is also a relativistic correction to the elliptical shape that explains the perihelion precession of Mercury, for example.
We therefore provide here an alternative derivation of the orbital equation for a massive particle in Schwarzschild geometry, which gives a solution that brings us closer to the original formula for an ellipse.
This is:

$$\boldsymbol{r}(\emptyset) = \frac{\boldsymbol{a}\left(\mathbf{1}-\boldsymbol{e}^{\mathbf{2}}\right)}{\mathbf{1}+\boldsymbol{ecos}[\emptyset-\theta]}$$

This equation is compared with the relativistic result (see equation_3_5_5a) at the end of this chapter:

$$\boldsymbol{r} = \frac{\boldsymbol{a}\left(\mathbf{1}-\boldsymbol{e}^{\mathbf{2}}\right)}{\mathbf{1}+\boldsymbol{ecos}[\emptyset-\epsilon\emptyset]}$$

Here we see that θ is not a constant but a function of $\emptyset$ and changes by a factor of ϵ.

From "*General Relativity an introduction for Physics*" by M.P. Hobson, G. Efstathlou, and A.N. Lasenby Page 230 (M.P Hobson, 2006) .

We limit ourselves to the equatorial plane $\theta = \pi/2$, so that the Schwarzschild metric for a massive particle reduces to:

$$c^2 d\tau^2 = c^2\left(1-\frac{2GM}{c^2 r}\right)dt^2 - \left(1-\frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\emptyset^2$$

The metric equation then becomes:

$$c^2\left(1-\frac{2GM}{c^2 r}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1-\frac{2GM}{c^2 r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\emptyset}{d\tau}\right)^2 = c^2$$

After multiplying by $\left(1-\frac{2GM}{c^2 r}\right)$, the following results:

$$c^2\left(1-\frac{2GM}{c^2 r}\right)^2\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\emptyset}{d\tau}\right)^2\left(1-\frac{2GM}{c^2 r}\right) = c^2\left(1-\frac{2GM}{c^2 r}\right)$$

Rearrangement:

$$\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\emptyset}{d\tau}\right)^2\left(1-\frac{2GM}{c^2 r}\right) - c^2\left(1-\frac{2GM}{c^2 r}\right)^2\left(\frac{dt}{d\tau}\right)^2 = c^2\left(\frac{2GM}{c^2 r}-1\right)$$

We substitute the retained quantities:

$$\left(1-\frac{2GM}{c^2 r}\right)\frac{dt}{d\tau} = \frac{E}{c^2}$$

$$r^2\frac{d\emptyset}{d\tau} = L$$

We then obtain:

$$\left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{r^2}\left(1-\frac{2GM}{c^2r}\right) - \frac{E^2}{c^2} = c^2\left(\frac{2GM}{c^2r}-1\right) = \frac{2GM}{r} - c^2$$

$$\left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{r^2}\left(1-\frac{2GM}{c^2r}\right) - \frac{2GM}{r} = c^2\left(\frac{E^2}{c^4}-1\right)$$

$$\left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{r^2} = c^2\left(\frac{E^2}{c^4}-1\right) + \frac{2GM}{r} + \frac{2GML^2}{c^2r^3}$$

Now:

$$\frac{dr}{d\tau} = \frac{dr}{d\emptyset}\frac{d\emptyset}{d\tau} = \frac{L}{r^2}\frac{dr}{d\emptyset}$$

Filling this into the previous equation:

$$\left(\frac{L}{r^2}\frac{dr}{d\emptyset}\right)^2 + \frac{L^2}{r^2} = c^2\left(\frac{E^2}{c^4}-1\right) + \frac{2GM}{r} + \frac{2GML^2}{c^2r^3}$$

Divide by L^2:

$$\left(\frac{1}{r^2}\frac{dr}{d\emptyset}\right)^2 + \frac{1}{r^2} = \frac{c^2}{L^2}\left(\frac{E^2}{c^4}-1\right) + \frac{2GM}{rL^2} + \frac{2GM}{c^2r^3}$$

Now replace with $u = 1/r$

$$\frac{du}{d\emptyset} = \frac{du}{dr}\frac{dr}{d\emptyset} = \frac{-1}{r^2}\frac{dr}{d\emptyset} => \frac{1}{r^2}\frac{dr}{d\emptyset} = -\frac{du}{d\emptyset}$$

Now the equation becomes:

$$\left(\frac{du}{d\emptyset}\right)^2 + u^2 = \frac{c^2}{L^2}\left(\frac{E^2}{c^4}-1\right) + \frac{2GMu}{L^2} + \frac{2GMu^3}{c^2}$$

We differentiate this equation with respect to$\emptyset$ to obtain:

$$2\frac{du}{d\emptyset}\frac{d^2u}{d\emptyset^2} + 2u\frac{du}{d\emptyset} = \frac{2GM}{L^2}\frac{du}{d\emptyset} + \frac{6GMu^2}{c^2}\frac{du}{d\emptyset}$$

We divide by $2\frac{du}{d\emptyset}$ (assuming that $\frac{du}{d\emptyset} \neq 0$) :

$$\frac{d^2u}{d\emptyset^2} + u = \frac{GM}{L^2} + \frac{3GMu^2}{c^2} \qquad (44)$$

If we ignore the last term for now, we get the equation according to Newtonian theory, whose solution is:

$$u = \frac{GM}{L^2}(1 + e\cos\emptyset) \quad or \quad r = \frac{L^2}{GM(1+e\cos\emptyset)} \qquad (45)$$

This describes an ellipse, where the parameter *e* represents the *eccentricity* of the orbit. For example, we can draw the orbit of a planet around the sun. We can write the distance to the closest point (*perihelion*) as $r_1 = a(1-e)$ and the distance to the furthest point (*aphelion*) as $r_2 = a(1+e)$.

Derived from (45) and again substituted with r=1/u gives:

$$r = \frac{L^2}{GM(1 + e\cos\emptyset)} \quad => \quad r_{max} = \frac{L^2}{GM(1-e)} \quad \text{and } r_{min} = \frac{L^2}{GM(1+e)}$$

The semi-major axis *a* is then given by:

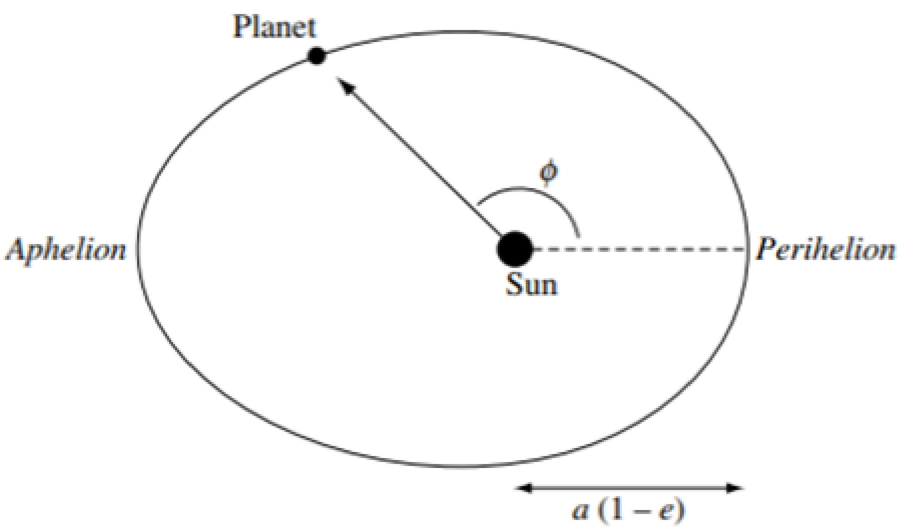
$$\mathrm{a} = \frac{r_{max} + r_{min}}{2} = \frac{L^2}{2GM}\left(\frac{1}{(1-e)} + \frac{1}{(1+e)}\right) = \frac{L^2}{2GM}\left(\frac{1+e+1-e}{(1-e)(1+e)}\right)$$

So the equation of motion requires that the semi-major axis is given by:

$$\boldsymbol{a} = \frac{\boldsymbol{L^2}}{\boldsymbol{GM}(\mathbf{1} - \boldsymbol{e}^{\mathbf{2}})} \tag{46}$$

Therefore:

$$r_{max} = \frac{L^2}{GM(1-e)} = a(1+e) \quad \text{and } r_{min} = \frac{L^2}{GM(1+e)} = a(1-e)$$



The elliptical orbit of a planet around the sun; *e* is the eccentricity of the orbit.

Now, to include the third term (from equation 44), the solution becomes:

$$u = \frac{GM}{L^2}(1 + e\cos\emptyset) + \Delta\mathrm{u} \tag{47}$$

$$\frac{du}{d\emptyset} = -\frac{GM}{L^2}e\sin\emptyset + \frac{d\Delta u}{d\emptyset}$$

$$\frac{d^2u}{d\emptyset^2} = -\frac{GM}{L^2}e\cos\emptyset + \frac{d^2\Delta u}{d\emptyset^2}$$

We substitute this into formula (44):

$$\frac{d^2u}{d\emptyset^2} + u = \frac{GM}{L^2} + \frac{3GMu^2}{c^2} \tag{44}$$

$$\frac{d^2u}{d\emptyset^2} + u = \frac{GM}{L^2}(1 + e\cos\emptyset - e\cos\emptyset) + \frac{d^2\Delta u}{d\emptyset^2} + \Delta\mathrm{u} = \frac{GM}{L^2} + \frac{d^2\Delta u}{d\emptyset^2} + \Delta\mathrm{u}$$

$$\frac{d^2\Delta u}{d\emptyset^2}+\Delta\mathrm{u}=-\frac{GM}{L^2}+\frac{d^2u}{d\emptyset^2}+u=-\frac{GM}{L^2}+\frac{GM}{L^2}+\frac{3GMu^2}{c^2}=\frac{3GMu^2}{c^2}$$

$$\frac{d^2\Delta u}{d\emptyset^2}+\Delta\mathrm{u}=\frac{3GM}{c^2}\left(\left(\boldsymbol{\frac{GM}{L^2}}\right)^{\mathbf{2}}+\left(\boldsymbol{\frac{GM}{L^2}e}\,\mathbf{cos}\,\emptyset\right)^{\mathbf{2}}+(\Delta\mathrm{u})^2+\mathbf{2}\left(\boldsymbol{\frac{GM}{L^2}}\right)^{\mathbf{2}}\boldsymbol{e}\,\mathbf{cos}\,\emptyset+2\frac{GM}{L^2}\Delta\mathrm{u}+2\frac{GM}{L^2}e\cos\emptyset.\Delta\mathrm{u}\right)$$

We find that, to first order in Δu ,

$$\frac{d^2\Delta u}{d\emptyset^2}+\Delta\mathrm{u}=\frac{3(GM)^3}{c^2L^4}\left(\mathbf{1}+(\boldsymbol{e}\,\mathbf{cos}\,\emptyset)^{\mathbf{2}}+\mathbf{2}\boldsymbol{e}\,\mathbf{cos}\,\emptyset\right)$$

A particular solution of the equation is:

$$\boldsymbol{\Delta\mathrm{u}=\frac{3(GM)^3}{c^2L^4}\left[1+e^2\left(\frac{1}{2}-\frac{1}{6}\cos 2\,\emptyset\right)+\mathrm{e}\,\emptyset\sin\emptyset\right]} \qquad (\mathbf{48})$$

This can be verified by direct differentiation of (48):

$$\frac{d\Delta u}{d\emptyset}=\frac{3(GM)^3}{c^2L^4}\left[\frac{1}{3}e^2\sin 2\,\emptyset+\mathrm{e}\sin\emptyset+\mathrm{e}\,\emptyset\cos\emptyset\right]$$

$$\frac{d^2\Delta u}{d\emptyset^2}=\frac{3(GM)^3}{c^2L^4}\left[\frac{2}{3}e^2\cos 2\,\emptyset+\mathrm{e}\cos\emptyset+\mathrm{e}\cos\emptyset-\mathrm{e}\,\emptyset\sin\emptyset\right]$$

$$\frac{d^2\Delta u}{d\emptyset^2}=\frac{3(GM)^3}{c^2L^4}\left[\frac{2}{3}e^2\cos 2\,\emptyset+2\mathrm{e}\cos\emptyset-\mathrm{e}\,\emptyset\sin\emptyset\right]$$

Substituting into (48):

$$\frac{d^2\Delta u}{d\emptyset^2}+\Delta\mathrm{u}=\frac{3(GM)^3}{c^2L^4}\left[\frac{2}{3}e^2\cos 2\,\emptyset+2\mathrm{e}\cos\emptyset-\mathrm{e}\,\emptyset\sin\emptyset+1+e^2\left(\frac{1}{2}-\frac{1}{6}\cos 2\,\emptyset\right)+\mathrm{e}\,\emptyset\sin\emptyset\right]$$

$$\frac{d^2\Delta u}{d\emptyset^2}+\Delta\mathrm{u}=\frac{3(GM)^3}{c^2L^4}\left[1+\frac{1}{2}e^2+\frac{1}{2}e^2\cos 2\,\emptyset+2\mathrm{e}\cos\emptyset\right]$$

$$\frac{d^2\Delta u}{d\emptyset^2}+\Delta\mathrm{u}=\frac{3(GM)^3}{c^2L^4}\left[1+\frac{1}{2}e^2(1+\cos 2\,\emptyset)+2\mathrm{e}\cos\emptyset\right]$$

$$\frac{d^2\Delta u}{d\emptyset^2}+\Delta\mathrm{u}=\frac{3(GM)^3}{c^2L^4}\left[1+\frac{1}{2}e^2(\sin^2\emptyset+\cos^2\emptyset+\cos^2\emptyset-\sin^2\emptyset)+2\mathrm{e}\cos\emptyset\right]$$

$$\frac{d^2\Delta u}{d\emptyset^2}+\Delta\mathrm{u}=\frac{3(GM)^3}{c^2L^4}[1+e^2\cos^2\emptyset+2\mathrm{e}\cos\emptyset]$$

So, equation (44) is correct.

Now we substitute Δu into equation (3):

$$u=\frac{GM}{L^2}(1+e\cos\emptyset)+\Delta\mathrm{u}=\frac{GM}{L^2}(1+e\cos\emptyset)+\frac{3(GM)^3}{c^2L^4}\left[1+e^2\left(\frac{1}{2}-\frac{1}{6}\cos 2\,\emptyset\right)+\mathrm{e}\,\emptyset\sin\emptyset\right]$$

$$u=\frac{GM}{L^2}(1+e\cos\emptyset)+\frac{3(GM)^3}{c^2L^4}\mathrm{e}\,\emptyset\sin\emptyset+\frac{3(GM)^3}{c^2L^4}\left[1+e^2\left(\frac{1}{2}-\frac{1}{6}\cos 2\,\emptyset\right)\right]$$

Since the constant $\frac{3(GM)^3}{c^2L^4}$ is very small, the last three terms on the right-hand side of the equation can be neglected—they are so small that they have no noticeable influence and are therefore not useful for testing the theory.

However, the last term $e\frac{3(GM)^3}{c^2L^4}\emptyset\sin\emptyset$ is a special case. Although this term is initially small, it grows slowly as Ø increases, because Ø itself increases over time. As a result, the effect accumulates, and we must retain this term.

$$u = \frac{GM}{L^2}\left[1 + e\left(\cos\emptyset + \frac{3(GM)^2}{c^2L^2}\emptyset\sin\emptyset\right)\right] + \frac{3(GM)^3}{c^2L^4}\left[1 + e^2\left(\frac{1}{2} - \frac{1}{6}\cos 2\emptyset\right)\right]$$

So, our approximate solution is:

$$u = \frac{GM}{L^2}\left[1 + e\left(\cos\emptyset + \frac{3(GM)^2}{c^2L^2}\emptyset\sin\emptyset\right)\right]$$

Using the relationship:

$$cos\left[\emptyset\left(1 - \frac{3(GM)^2}{c^2L^2}\right)\right] = cos\left(\emptyset - \frac{3(GM)^2}{c^2L^2}\emptyset\right) = \cos\emptyset\cos\frac{3(GM)^2}{c^2L^2}\emptyset + \sin\emptyset\sin\frac{3(GM)^2}{c^2L^2}\emptyset$$

$$\approx \cos\emptyset + \frac{3(GM)^2}{c^2L^2}\emptyset\sin\emptyset \qquad \text{for} \quad \frac{3(GM)^2}{c^2L^2} \ll 1\,,$$

we can now write:

$$u \approx \frac{GM}{L^2}\left\{1 + ecos\left[\emptyset\left(1 - \frac{3(GM)^2}{c^2L^2}\right)\right]\right\} = \frac{GM}{L^2}\{1 + ecos[\emptyset(1-\epsilon)]\}$$

For r=1/u, we get:

$$r = \frac{L^2}{GM\{1 + ecos[\emptyset(1-\epsilon)]\}} \tag{5}$$

Here:

$$\epsilon = \frac{3(GM)^2}{c^2L^2}$$

This expression shows that the orbit is indeed periodic, but with a period of $2\pi/(1-\epsilon)$. This means that the values of *r* repeat at an angle greater than 2π . As a result, the orbit does not close perfectly as in a classical ellipse: the ellipse slowly rotates around the focus. We call this phenomenon precession (see figure below).

After each complete orbit, the ellipse is slightly rotated around the focal point, by an angle of:

$$\Delta\emptyset = \frac{2\pi}{1-\epsilon} - 2\pi = \frac{2\pi\epsilon}{1-\epsilon} \approx 2\pi\epsilon = \frac{6\pi(GM)^2}{c^2L^2}$$

We replace *L* using equation (2):

$$a = \frac{L^2}{GM(1-e^2)} \tag{2}$$

By substituting this expression into equation (5):

$$r = \frac{L^2}{GM\{1 + ecos[\emptyset(1-\epsilon)]\}}$$

We obtain the following for the trajectory:

$$\boldsymbol{r} = \frac{\boldsymbol{a}(\boldsymbol{1} - \boldsymbol{e^2})}{\boldsymbol{1} + \boldsymbol{ecos}[\emptyset(\boldsymbol{1} - \epsilon)]} \tag{5a}$$

With:

$$\epsilon = \frac{3(GM)^2}{c^2L^2} \quad \text{or} \quad \epsilon = \frac{3(GM)^2}{c^2GMa(1-e^2)} = \frac{3GM}{c^2a(1-e^2)}$$

Derived from Kepler's third law:

$$T^2 = \frac{4\pi^2a^3}{G(M+m)} \approx \frac{4\pi^2a^3}{GM} => T = 2\pi a\sqrt{\frac{a}{GM}}$$

For the velocity v :

$$v = \frac{L}{r\cos\alpha} = \frac{\sqrt{aGM(1-e^2)}}{\boldsymbol{a}(\boldsymbol{1} - \boldsymbol{e^2})} \frac{(\boldsymbol{1} + \boldsymbol{ecos}[\emptyset(\boldsymbol{1} - \epsilon)])}{\cos\alpha}$$

Therefore:

$$v = \sqrt{\frac{GM}{a(1-e^2)}} \frac{(\boldsymbol{1} + \boldsymbol{ecos}[\emptyset(\boldsymbol{1} - \epsilon)])}{\cos\alpha}$$

where:

$$\epsilon = \frac{3(GM)^2}{c^2L^2} = \frac{3(GM)^2}{c^2aGM(1-e^2)} = \frac{3GM}{c^2a(1-e^2)}$$

Because:

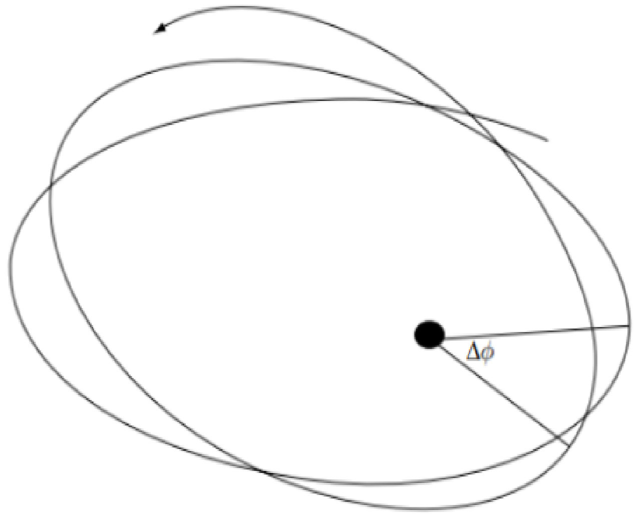
$$L^2 = aGM(1-e^2)$$

Filled in gives:

$$\Delta\emptyset = \frac{6\pi(GM)^2}{c^2aGM(1-e^2)}$$

Ultimately, we obtain the following for the precession angle:

$$\boldsymbol{\Delta}\emptyset = \frac{\boldsymbol{6\pi GM}}{\boldsymbol{a}(\boldsymbol{1} - \boldsymbol{e^2})\boldsymbol{c^2}} \tag{6}$$



Precession of an elliptical orbit (greatly exaggerated)

We apply equation (6) to the orbit of Mercury, with the following parameters:

- period = 88 days
- a=5.8x10$^{(10)\ m}$
- e=0.2
- M_s =2x10^{30} kg

we find:

$$T = \sqrt{\frac{4\pi^2 a^3}{GM}} = 87.95\ days$$

$$\Delta\emptyset = \frac{6\pi GM}{a(1-e^2)c^2} = 5.02 * 10^{-7}\ \text{ rad per cycle}$$

To calculate the precession per century:

$$\Delta\emptyset = 5.02 * 10^{-7}\ * \left(100 * \frac{365.25}{88}\right) * \left(\frac{360 * 60 * 60}{2\pi}\right)$$

$$\Delta\emptyset = \mathbf{43}^{''}\ \text{(arc seconds per century).}$$

In reality, the measured precession is:

$$5599^{''}.7 \pm 0.4^{''}\ \text{ per century}$$

The vast majority of this is caused by gravitational influences from other planets. But after correcting for these disturbances, there remains a residual deviation that corresponds surprisingly well with the prediction of general relativity. For other celestial bodies, we find similar results (in arc seconds per century):

Object	Observed residual precession	Predicted residual precession
Mercury	43.1+/-0.5	43.03
Venus	8+/-5	8.6
Earth	5	3.8
Icarus	10+/-1	10.3

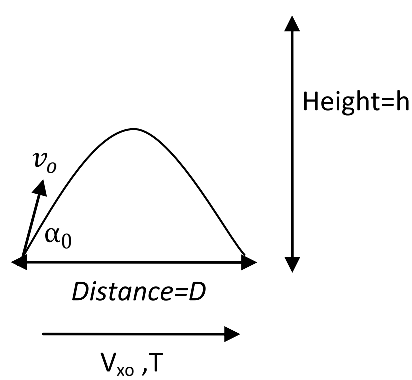
The results therefore correspond perfectly with the predictions of general relativity theory. Einstein added this calculation for Mercury to his 1915 article on general relativity. In doing so, he immediately solved one of the major outstanding problems in classical celestial mechanics—an impressive first test of his new, complex theory. You can imagine how much confidence this gave him in its accuracy.

4.8 Calculation of a Spherical Orbit

As an exercise, we are interested in calculating the trajectory of a ball using the rules of general relativity, as opposed to the classical (Newtonian) approach.
For the General Relativity approach, we assume that the trajectory of the bullet is forced by the mass of the Earth to follow an elliptical shape. For the calculation, we use the Schwarzschild equation. But first, we start with the Newtonian approach.

4.8.1 Newtonian Approach

We consider a ball fired at an angle, with a horizontal distance *D* between the starting point and the target, and a maximum height *h*. The gravitational acceleration is *g*, and the initial velocity of the ball has components v_{xo} (horizontal) and v_{yo} (vertical).



a) Time and velocity components

The time required for the bullet to travel the distance *D* at a constant horizontal velocity v_{x0} is:

$$v_{x0} = \frac{D}{T} \; => T = \frac{D}{v_{x0}}$$

To cover the distance *D*, the ball also needs an upward velocity, otherwise it will hit the ground too early. This requires an initial velocity component in the y-direction v_{y0} . This velocity is determined by the horizontal distance *D* and the time *T*. So, *T* is also the time it takes to go up from the ground and fall back to the ground.

Because the motion is symmetrical, the time to reach the highest point is:

$$T_{upwards} \; = \frac{T}{2}$$

At this point, the vertical velocity is zero. From the equation of motion, we then get:

$$v_y = v_{y0} - gt = 0$$

at:

$$t = \frac{T}{2}$$

$$=> v_{y0} = gt = g\frac{T}{2} = g\frac{D}{2v_{x0}}$$

b) Height and relationship with time

The height is reached at time T/2, so:

$$h = v_{y0} \cdot \frac{T}{2} - \frac{1}{2}g\left(\frac{T}{2}\right)^2 = \frac{gD}{2v_{x0}} \cdot \frac{\mathrm{D}}{2v_{x0}} - \frac{1}{2}g\left(\frac{\mathrm{D}}{2v_{x0}}\right)^2 = \frac{g\mathrm{D}^2}{8v_{x0}^2}$$

Conversely, this gives:

$$v_{xo} \; = D\sqrt{\frac{g}{8h}}$$

When the ball falls back from the highest point *h*, it takes T/2 to reach the ground:

c) Total speed and trajectory equation

To reach the highest point:

$$v_{y0} = g\frac{T}{2} \; = g\sqrt{\frac{2h}{g}} = \sqrt{2hg}$$

The total initial velocity:

$${v_0}^2 = v_{x0}^2 + v_{y0}^2 = \frac{gD^2}{8h} + 2hg = g\left(\frac{D^2 + 16h^2}{8h}\right)$$

Is therefore:

$$v_0 = \sqrt{g\left(\frac{D^2 + 16h^2}{8h}\right)}$$

$$v_{x0} = v_0 \cos \alpha_0$$

The angle of inclination α_0 at which the ball is fired follows from:

$$\tan\alpha_0 = \frac{v_{y0}}{v_{x0}} = \frac{\sqrt{2hg}}{D\sqrt{\frac{g}{8h}}} = \frac{4h}{D}$$

Therefore:

$$\tan\alpha_0 = \frac{4h}{D}$$

d) Trajectory equation

The y-position as a function of time:

$$y(t) = v_{y0}t - \frac{1}{2}gt^2 = g\frac{T}{2}t - \frac{1}{2}gt^2 = \frac{1}{2}gt(T-t)$$

$$y(t) = \frac{1}{2}\mathrm{gt}\left(\frac{D}{v_{x0}} - \mathrm{t}\right)$$

In terms of the x-position:

$$x = v_{x0}\mathrm{t} \Rightarrow \mathrm{t} = \frac{\mathrm{x}}{v_{x0}} \Rightarrow y(x) = \frac{1}{2}\mathrm{g}\cdot\frac{\mathrm{x}}{v_{x0}}\left(\frac{D}{v_{x0}} - \frac{\mathrm{x}}{v_{x0}}\right) = \frac{1}{2}\frac{\mathrm{g}}{v_{x0}^2}\mathrm{x(D-x)}$$

The trajectory of the ball is therefore a parabola:

$$y(x) = \frac{1}{2}\frac{\mathrm{g}}{v_{x0}^2}\mathrm{x(D-x)}$$

This is therefore a function of the required distance *D* when the initial horizontal velocity component is v_{x0} .

e) Sample calculation

Horizontal distance *D* (m)	$(v_{xo}\ \mathrm{m/s})$	**Time *T* (s)**	**Height *h* (m)**	**Total velocity** $(v_{xo}\ \mathrm{m/s})$
10	5	2	4.93	11.06
10	500	0.02	0.000493	500
100	5	20	493	99
100	50	2	4.93	51

(where g=9.87 m/s^2)

f) Next step

Now that we have fully developed the Newtonian approximation, we can compare it with the calculation based on Schwarzschild geometry within general relativity. This comparison follows in the next section.

4.8.2 Schwarzschild Approach

For this approximation, we consider the spherical orbit as part of an ellipse with the center of the Earth as one of the foci. We use the results from the Schwarzschild equation in section 4.7**4.7Alternative Derivation of the Orbital Equation**andWe obtain the following for the trajectory:

The semi-major axis is:

$$a = \frac{L^2}{GM(1-e^2)} \tag{2}$$

The parameter e represents the eccentricity of the bullet's trajectory. The perihelion is $r_1 = a(1-e)$ and the aphelion is $r_2 = a(1+e)$.

$$e = \sqrt{1-\frac{b^2}{c^2}} = \frac{r_2 - r_1}{r_2 + r_1}$$

So for a circle, e=0 and $r = r_1 = r_2$ =a.

To obtain an ellipse, as in the drawing below, where the center of the Earth coincides with the left focus of the ellipse, the equation looks like this:

$$\boldsymbol{r}(\emptyset) = \frac{\boldsymbol{a}\left(\mathbf{1}-\boldsymbol{e}^{\mathbf{2}}\right)}{\mathbf{1}-\boldsymbol{ecos}[\emptyset(\mathbf{1}-\epsilon)]} \tag{2a}$$

Now we will derive the angle α between v , the velocity tangential to the ellipse, and v_{per} , perpendicular to *r*, to determine the angular momentum. In this experiment, v is the total velocity of the bullet along the ellipse, while v_{per} is the component of the velocity v relative to the Earth's surface and, as mentioned, perpendicular to $r(\emptyset)$.

$$\tan\alpha = \frac{dr}{rd\emptyset} = \frac{\{1-ecos[\emptyset(1-\epsilon)]\}}{a(1-e^2)} \frac{\{a(1-e^2)(1-\epsilon)(esin[\emptyset(1-\epsilon)])\}}{\{1-ecos[\emptyset(1-\epsilon)]\}^2}$$

$$\tan\alpha = \frac{dr}{rd\emptyset} = \frac{e(1-\epsilon)sin[\emptyset(1-\epsilon)]}{1-ecos[\emptyset(1-\epsilon)]}$$

$$\boldsymbol{\alpha} = \boldsymbol{arctan}\left\{\frac{\boldsymbol{e}(\mathbf{1}-\boldsymbol{\epsilon})\boldsymbol{sin}[\emptyset(\mathbf{1}-\boldsymbol{\epsilon})]}{\mathbf{1}-\boldsymbol{ecos}[\emptyset(\mathbf{1}-\boldsymbol{\epsilon})]}\right\}$$

If:

$$\cos\alpha = \frac{1}{\sqrt{1+\tan\alpha}}$$

Then we get:

$$\cos\alpha = \left[1+\left(\frac{e(1-\epsilon)sin[\emptyset(1-\epsilon)]}{1-ecos[\emptyset(1-\epsilon)]}\right)^2\right]^{-\frac{1}{2}}$$

$$\cos\alpha = \left[\frac{1-2ecos[\emptyset(1-\epsilon)]+e^2cos^2[\emptyset(1-\epsilon)]+\{e(1-\epsilon)sin[\emptyset(1-\epsilon)]\}^2}{\{1-ecos[\emptyset(1-\epsilon)]\}^2}\right]^{-\frac{1}{2}}$$

Because of the negative root sign, we reverse the equation:

$$\cos\alpha = \frac{1 - ecos[\emptyset(1-\epsilon)]}{\left[1 - 2ecos[\emptyset(1-\epsilon)] + e^2cos^2[\emptyset(1-\epsilon)] + (1 - 2\epsilon + \epsilon^2)e^2sin^2[\emptyset(1-\epsilon)]\right]^{1/2}}$$

$$\cos\alpha = \frac{1 - ecos[\emptyset(1-\epsilon)]}{\left[1 - 2ecos[\emptyset(1-\epsilon)] + e^2cos^2[\emptyset(1-\epsilon)] + e^2sin^2[\emptyset(1-\epsilon)] - \epsilon(2-\epsilon)e^2sin^2[\emptyset(1-\epsilon)]\right]^{\frac{1}{2}}}$$

$$\boldsymbol{\cos\alpha = \frac{1 - ecos[\emptyset(1-\epsilon)]}{[1 - 2ecos[\emptyset(1-\epsilon)] + e^2(1 - \epsilon(2-\epsilon)sin^2[\emptyset(1-\epsilon)])]^{1/2}}} \qquad (2b)$$

The momentum *L* is constant across the entire ellipse. The momentum is the velocity perpendicular to *r,* multiplied by *r* (assuming that the mass is unity):

$$L = v_{per} \cdot r = v \cdot \cos\alpha \cdot r$$

So here, the following applies:

$$\boldsymbol{L = v_{x0} \cdot R_{earth}}$$

According to equation (2):

$$L = \sqrt{\mathrm{a}GM(1-e^2)}$$

The velocity *v* is given by:

$$v = \frac{L}{r\cos\alpha} = \frac{\left(aGM(1-e^2)\right)^{1/2}}{a(1-e^2)\cos\alpha}(1 - ecos[\emptyset(1-\epsilon)])$$

This simplifies to:

$$v = \left(\frac{GM}{a(1-e^2)}\right)^{\frac{1}{2}} \frac{(1 - ecos[\emptyset(1-\epsilon)])}{\cos\alpha} \qquad (2c)$$

Substitute $\cos(\alpha)$ from equation (2b) into equation (2c):

$$v = \left(\frac{GM}{a(1-e^2)}\right)^{\frac{1}{2}} \frac{(1 - ecos[\emptyset(1-\epsilon)])}{1 - ecos[\emptyset(1-\epsilon)]} [1 - 2ecos[\emptyset(1-\epsilon)] + e^2(1 - \epsilon(2-\epsilon)sin^2[\emptyset(1-\epsilon)])]^{1/2}$$

The instantaneous velocity as a function of Ø is:

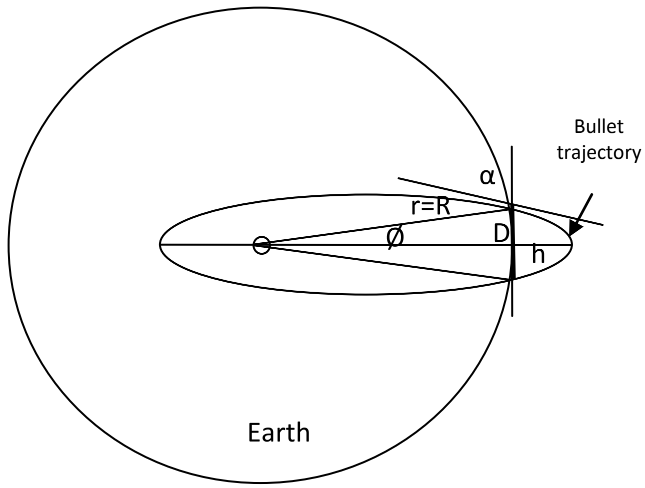
$$\boldsymbol{v = \left(\frac{GM}{a(1-e^2)}\left(1 - 2ecos[\emptyset(1-\epsilon)] + e^2\{1 - \epsilon(2-\epsilon)sin^2[\emptyset(1-\epsilon)]\}\right)\right)^{\frac{1}{2}}} \qquad \boldsymbol{(2d)}$$

Obtained from the previous chapter ϵ :

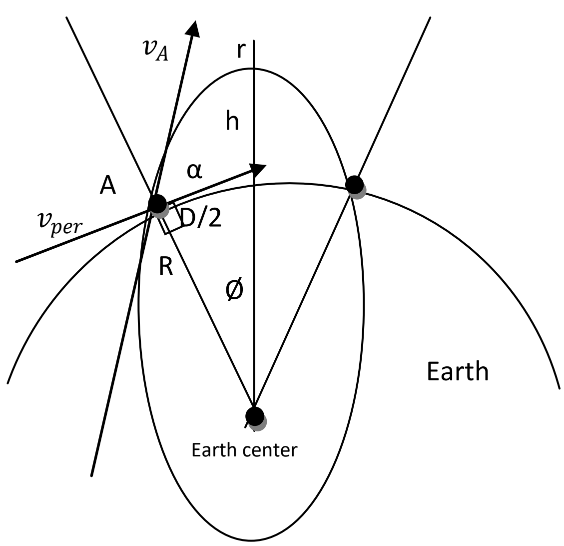
$$\epsilon = \frac{3(GM)^2}{c^2L^2} = \frac{3(GM)^2}{c^2aGM(1-e^2)} = \frac{3GM}{c^2a(1-e^2)}$$

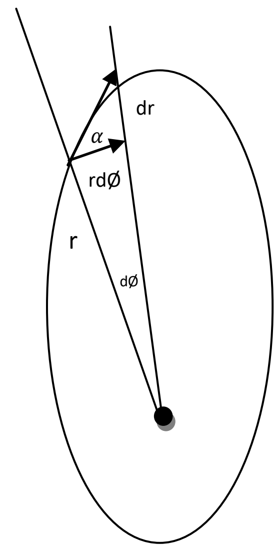
Here,

$$\boldsymbol{\epsilon} = \frac{3(GM)^2}{c^2L^2} = \frac{3(GM)^2}{c^2(v_{x0}R_{earth})^2} = \boldsymbol{\frac{3c^2}{v_{x0}{}^2}\left(\frac{GM}{c^2R_{earth}}\right)^2} \text{ this has no dimension} \quad (2e)$$



To zoom in a little further:





$$ØR = \frac{D}{2} => \ Ø = \frac{D}{2R}$$

$$v_{per} = v_{x\mathrm{A}} = v\cos(\alpha) \text{ en } v_{y\mathrm{A}} = v\sin(\alpha)$$

From (2a)

$$a(1-e^2) = r\{1 - ecos[Ø(1-\epsilon)]\}$$

From (2d)

$$v = \left(\frac{GM}{a(1-e^2)}(1 - 2ecos[Ø(1-\epsilon)] + e^2\{1 - \epsilon(2-\epsilon)sin^2[Ø(1-\epsilon)]\})\right)^{\frac{1}{2}}$$

$$v^2 = \frac{GM}{a(1-e^2)}(1 - 2ecos[Ø(1-\epsilon)] + e^2\{1 - \epsilon(2-\epsilon)sin^2[Ø(1-\epsilon)]\})$$

$$v^2 = GM\frac{(1 - 2ecos[Ø(1-\epsilon)] + e^2\{1 - \epsilon(2-\epsilon)sin^2[Ø(1-\epsilon)]\})}{r\{1 - ecos[Ø(1-\epsilon)]\}}$$

$$\frac{v^2 r}{GM}\{1-ecos[\emptyset(1-\epsilon)]\} = 1-2ecos[\emptyset(1-\epsilon)] + e^2\{1-\epsilon(2-\epsilon)sin^2[\emptyset(1-\epsilon)]\}$$

$$\boldsymbol{e^2}\{1-\epsilon(2-\epsilon)sin^2[\emptyset(1-\epsilon)]\} - \boldsymbol{e}cos[\emptyset(1-\epsilon)]\left(2-\frac{v^2 r}{GM}\right) + \left(1-\frac{v^2 r}{GM}\right) = 0$$

$$e = \frac{cos[\emptyset(1-\epsilon)]\left(2-\frac{v^2 r}{GM}\right) \pm \sqrt{\left[cos[\emptyset(1-\epsilon)]\left(2-\frac{v^2 r}{GM}\right)\right]^2 - 4\left(1-\frac{v^2 r}{GM}\right)\{1-\epsilon(2-\epsilon)sin^2[\emptyset(1-\epsilon)]\}}}{2\{1-\epsilon(2-\epsilon)sin^2[\emptyset(1-\epsilon)]\}}$$

For the starting point at the intersection of the Earth and the orbit, *r=R*. (*R* is the radius of the Earth) and $\emptyset = \frac{D}{2R}$.

From (2a):

$$r = \frac{a(1-e^2)}{1-ecos[\emptyset(1-\epsilon)]}$$

$$a(1-e^2) = R\left\{1-ecos\left[\frac{D}{2R}(1-\epsilon)\right]\right\}$$

$$a = \frac{R\left\{1-ecos\left[\frac{D}{2R}(1-\epsilon)\right]\right\}}{(1-e^2)} \tag{3}$$

$$\boldsymbol{e} = \frac{\boldsymbol{cos}\left[\frac{\boldsymbol{D}}{\boldsymbol{2R}}(\boldsymbol{1}-\boldsymbol{\epsilon})\right]\left(\boldsymbol{2}-\frac{\boldsymbol{v^2R}}{\boldsymbol{GM}}\right) \pm \sqrt{\left[\boldsymbol{cos}\left[\frac{\boldsymbol{D}}{\boldsymbol{2R}}(\boldsymbol{1}-\boldsymbol{\epsilon})\right]\left(\boldsymbol{2}-\frac{\boldsymbol{v^2R}}{\boldsymbol{GM}}\right)\right]^{\boldsymbol{2}} - \boldsymbol{4}\left(\boldsymbol{1}-\frac{\boldsymbol{v^2R}}{\boldsymbol{GM}}\right)\{\boldsymbol{1}-\boldsymbol{\epsilon}(\boldsymbol{2}-\boldsymbol{\epsilon})\boldsymbol{sin^2}[\emptyset(\boldsymbol{1}-\boldsymbol{\epsilon})]\}}}{\boldsymbol{2}\{\boldsymbol{1}-\boldsymbol{\epsilon}(\boldsymbol{2}-\boldsymbol{\epsilon})\boldsymbol{sin^2}[\emptyset(\boldsymbol{1}-\boldsymbol{\epsilon})]\}}$$

Or here from equations (2), (2e), and (3):

$$R\left\{1-ecos\left[\frac{D}{2R}(1-\epsilon)\right]\right\} = \mathrm{a}(1-e^2) = \frac{L^2}{GM}$$

$$\mathrm{e} = \frac{1-\frac{L^2}{RGM}}{cos\left[\frac{D}{2R}(1-\epsilon)\right]} = \frac{1-\frac{L^2}{RGM}}{cos\left[\frac{D}{2R}\left(1-\frac{3c^2}{{v_{x0}}^2}\left(\frac{GM}{c^2 R_{eart\,h}}\right)^2\right)\right]} = \frac{1-\frac{(v_{x0}R)^2}{RGM}}{cos\left[\frac{D}{2R}\left(1-\frac{3c^2}{{v_{x0}}^2}\left(\frac{GM}{c^2 R}\right)^2\right)\right]}$$

$$\mathrm{e} = \frac{1-\frac{{v_{x0}}^2 R}{GM}}{cos\left[\frac{D}{2R}\left(1-\frac{3c^2}{{v_{x0}}^2}\left(\frac{GM}{c^2 R}\right)^2\right)\right]}$$

The given velocity at the point *r=R* is v. So for a given velocity, there are two solutions for *e*.

Here, *h* is the highest point of the ball's trajectory:

$$h = a(1+e) - R$$

Together with (3):

$$h = \frac{R\left\{1 - ecos\left[\frac{D}{2R}(1-\epsilon)\right]\right\}}{(1-e^2)}(1+e) - R = R\left\{\frac{1 - ecos\left[\frac{D}{2R}(1-\epsilon)\right]}{1-e} - 1\right\}$$

$$h = R\left\{\frac{1 - ecos\left[\frac{D}{2R}(1-\epsilon)\right] - 1 + e}{1-e}\right\} = R\frac{e\left(1 - cos\left[\frac{D}{2R}(1-\epsilon)\right]\right)}{1-e}$$

Here, D is the horizontal distance of the bullet on Earth, v is the initial velocity of the bullet, and R is the Earth's radius. As seen above, $\emptyset = \frac{D}{2R}$.

Or, pragmatically speaking, in our bullet example with v_{x0} en D as starting points:

$$h = a(1+e) - R = \frac{a(1-e^2)}{(1-e)} - R = \frac{L^2}{GM(1-e)} - \mathrm{R} = \frac{(v_{x0}R)^2}{GM(1-e)} - \mathrm{R}$$

$$h = \frac{(v_{x0}R)^2}{GM(1-e)} - \mathrm{R} = \frac{(v_{x0}R)^2}{GM(1-e)} - \mathrm{R}$$

Where:

$$\mathrm{e} = \frac{1 - \frac{{v_{x0}}^2\mathrm{R}}{GM}}{cos\left[\frac{D}{2R}\left(1 - \frac{3c^2}{{v_{x0}}^2}\left(\frac{GM}{c^2R}\right)^2\right)\right]}$$

Derivation of the circumference of an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x = acos\beta \text{ en } y = bsin\beta$$

$$Circumference = 4a\int_0^{\pi/2}\sqrt{\left(\frac{dx}{d\beta}\right)^2 + \left(\frac{dy}{d\beta}\right)^2}\,d\beta$$

$$= 4a\int_0^{\pi/2}\sqrt{a^2sin^2\beta + b^2cos^2\beta}\,d\beta$$

$$= 4a\int_0^{\pi/2}\sqrt{a^2(1-cos^2\beta) + b^2cos^2\beta}\,d\beta$$

$$= 4a\int_0^{\pi/2}\sqrt{a^2 - (a^2-b^2)cos^2\beta}\,d\beta$$

$$Circumference = 4a\int_0^{\pi/2}\sqrt{1 - e^2cos^2\beta}\,d\beta$$

There is no simple closed-form solution for the **circumference of an ellipse**. There are approximations, such as the Ramanujan approximation:

$$Circumference \approx \pi a\left[3\left(1 + \sqrt{1-e^2}\right) - \sqrt{10\sqrt{1-e^2} + 3(2-e^2)}\right]$$

Summary of the formulas used:

The starting points for this derivation are the speed of the ball along the Earth's surface ($v_{x0} = v_{per}$ loodrecht op r) and the required distance *D*. So, at the starting point where the ball is launched, we know the position and momentum of the ball and should be able to calculate its trajectory.

- $L = v_{x0} \cdot R_{earth}$ $\quad$ dus ϵ is L function of $L(v_{x0})$
- $\epsilon = \frac{3(GM)^2}{c^2 L^2}$ $\quad$ dus $\epsilon(v_{x0})$
- $\emptyset = \frac{D}{2R}$ $\quad$ dus $\emptyset(D)$
- $e = \frac{1-\frac{L^2}{RGM}}{\cos[\emptyset(1-\epsilon)]}$ $\quad$ dus $e(v_{x0}, D)$
- $\alpha = \arctan\left\{\frac{e(1-\epsilon)\sin[\emptyset(1-\epsilon)]}{1-e\cos[\emptyset(1-\epsilon)]}\right\}$ $\quad$ dus $e(v_{x0}, D)$
- $a = \frac{L^2}{GM(1-e^2)}$ $\quad$ dus $a(v_{x0}, D)$
- $h = a(1+e) - R$ $\quad$ dus $h(v_{x0}, D)$

Using these formulas, we obtain the results shown in the Excel table below:

Detailed results of calculations for the above example. The starting points are the (perpendicular to *r*) velocity of the bullet and the distance to be traveled.

	Newton				Schwarschild			
Vper0(m/s)	**5**	**50**	**500**	**1000**	**5**	**500**	**500**	**1000**
Distance (m)	**10**	**10**	**2000**	**2000**	**10**	**10**	**2000**	**2000**
Vr0(m/s)	9.87	0.10	19.73	9.87	9.76	0.10	19.66	9.71
speed (m/s)	11	500	500	1000	11	500	500	1000
epsilon					5.25E-03	5E-07	5.25E-07	1E-07
e(eccentricity)					1,000	0.996	0.996	0.984
a(m)					3.18E+06	3.18E+06	3.18E+06	3.20E+06
h(m)	4.93	4.93E-04	19.73	4.93	4.88	4.91E-04	19.66	4.85
alpha(rad)	1.10	0.000	0.04	0.010	1.10	0.000	0.04	0.01
alpha(deg)	63.13	0.0113	2.26	0.565	62.88	0.0113	2.25	0.556
Phi(rad)					7.87E-07	7.87E-07	1.57E-04	1.57E-04
L (angular momentum)	3.18E+07	3.18E+09	3.18E+09	6.36E+09	3.18E+07	3.18E+09	3.18E+09	6.36E+09
cos(alpha)	0.4520	1.0000	0.9992	1.0000	0.4558	1.0000	0.9992	1.0000
cos(alpha+phi)					0.4558	1.000	0.9992	1.000
Circ. (km)					12662	12894	12894	13346

3) **Analysis of the results**

Height differences

In the classical case, the maximum height of the projectile is $h \approx 4.93\ \mathrm{m}$ at low speeds. In the Schwarzschild approximation, this is slightly lower (e.g., 4.88 m), indicating a stronger effective gravity.

Eccentricity

An eccentricity of exactly 1 implies the classical parabola. The Schwarzschild approximation shows that the orbits are slightly elliptical with $e < 1$. For a horizontal velocity of 500 m/s, $e \approx 0.996$, while at 5 m/s, $e \approx 1$, which corresponds to a nearly parabolic orbit.

Directional angle

The deviation in direction angle Ø is very small at low speeds, but measurable at higher energies. For a projectile traveling at 500 m/s over 2 km, the deviation is $\emptyset \approx 1.57 * 10^{-4}\ \mathrm{rad}$, which corresponds to a precession of the ellipse axis.

Angular momentum and circumference

The angular momentum *L* increases with the initial velocity. The corresponding circumference of the elliptical trajectory (approximated) also shows an increase, reflecting the longer distance traveled by a powerful projectile.

4.8.3 Conclusion

- **Newtonian ballistics** is an excellent approximation for everyday situations.
- **Relativistic corrections** are subtle, but indispensable for highly accurate applications and at high speeds.
- **The Schwarzschild approximation** shows that even a simple bullet trajectory is, in principle, influenced by the curvature of space-time.

Part V – Coordinates and Formal Analysis

5 Coordinate Systems

5.1 Rectangular (Cartesian) Coordinate System

A coordinate system is created to distinguish between points in space. The most important features of a coordinate system are the **origin** and the **coordinate axes**. The origin can be chosen based on what is most practical, and a **Cartesian system** is usually chosen for the axes because its simplicity makes it mathematically easy to work with.

In a Cartesian coordinate system:

- The axes are perpendicular (orthogonal) to each other.
- The axes are independent of each other, i.e. changing the value of one coordinate does not affect the other.
- The axes have a direction and magnitude and can therefore be considered vectors.

A point in space is represented by its coordinates, for example A (x_a , y_a). The x_a can be found by drawing a line parallel to the y-axis; where that line intersects the x-axis is the point x_a . The same applies to the y_a .
The distance from point A to the origin can be found using Pythagoras' theorem.

$$(A - Origin)^2 = x_a^2 + y_a^2$$

If we work with a line segment between A and B, then the length is:

$$(A - B)^2 = (x_a - x_b)^2 + (y_a - y_b)^2$$

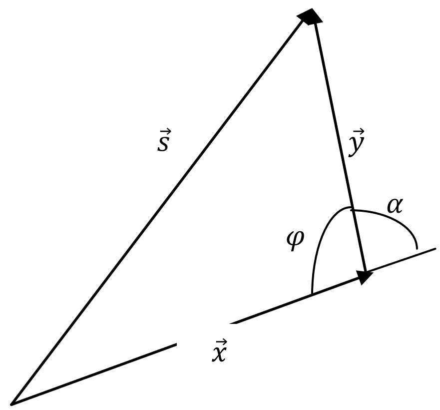
The advantage of this is that the length of the line segment is independent of the arbitrarily chosen origin; i.e., the values of x_a, y_a, x_b, y_b do change, but the difference $|A - B|$, which is the length of the line segment, does not change.

5.2 Non-Orthogonal Coordinate System

For practical reasons, a coordinate system can also be chosen whose axes are not orthogonal. In such a system, we can also describe positions and distances, but the calculations become slightly more complex.

A line segment $\vec{s}$ in this system is the sum of the basis vectors:

$$\vec{s} = \vec{x} + \vec{y}$$



The magnitude s of $\vec{s}$ can be found by the inner product of $\vec{s}$ with itself:

$$\vec{s}.\vec{s} = (\vec{x}+\vec{y}).(\vec{x}+\vec{y}) = \vec{x}.\vec{x}+\vec{x}.\vec{y}+\vec{y}.\vec{x}+\vec{y}.\vec{y}$$

$$s^2 = x^2 + (2\cos\alpha)xy + y^2$$

If φ is the angle between $\vec{x}$ en $\vec{y}$, thencos $\alpha = -\cos\varphi$, and therefore:

$$\cos\alpha = \cos(180^0 - \varphi) = -\cos\varphi$$

$$s^2 = x^2 + y^2 - 2xy\cos\varphi$$

This is the well-known **cosine rule**. So, in addition to the squares of the coordinates, the product of the coordinates is also part of the equation.

5.3 Curved Coordinates

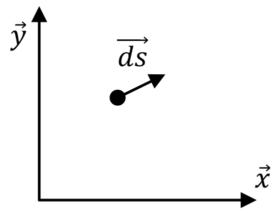
Instead of coordinate axes that are not orthogonal, it can also be practical to use curved coordinates. Working with these coordinates is obviously more complex, but Einstein used the following approach: A curved line can be considered as a line made up of infinitely small straight lines. By looking at an infinitely small area, these curved coordinates can be considered a local coordinate system with straight (linear) coordinates, which are not necessarily rectangular.

Because the coordinate system here involves infinitely small coordinates, the coordinates are denoted as *dx*, *dy,* and so on. In addition, these coordinates have coefficients, and these coefficients contain information about the curvature of the coordinate systems. In the case of curvature, these coefficients are no longer constants, but parameters that depend on their location along the coordinate systems.

It is said that gravity bends the coordinate systems and thus distorts space-time, creating a gravitational field and thereby causing acceleration. However, by choosing a curved coordinate system that moves and bends in the direction of the gravitational field, no force or gravity is experienced; thus, in the same way that in special relativity, a moving coordinate system was chosen to neutralize the speed of the moving object.

5.4 General Form for a Coordinate System

Let us derive an equation for the relationship between a line segment and its curved coordinate system.



As mentioned earlier, an infinitely small line segment d$\vec{s}$ is a vector, and its magnitude can be calculated as shown above:

$$\overrightarrow{ds}.\overrightarrow{ds} = \left(\overrightarrow{dx} + \overrightarrow{dy}\right).\left(\overrightarrow{dx} + \overrightarrow{dy}\right) = \overrightarrow{dx}.\overrightarrow{dx} + \overrightarrow{dx}.\overrightarrow{dy} + \overrightarrow{dy}.\overrightarrow{dx} + \overrightarrow{dy}.\overrightarrow{dy}$$

(for an orthogonal system)

To have a more general form (not necessarily orthogonal), it is assumed that each term has a coefficient $g_{\mu v}$:

$$ds^2 = g_{xx}\, dxdx + g_{xy}\, dxdy + g_{yx}\, dydx + g_{yy}\, dydy$$

Here, in the example of the cosine rule above, is:

$$g_{xx} = g_{yy} = 1 and \;\; g_{xy} = g_{yx} = -\cos\varphi$$

$g_{\mu v}$ is called the **metric tensor** and, in this two-dimensional coordinate system, can be considered as a matrix with 2x2 elements:

$$\begin{pmatrix} 1 & -\cos\varphi \\ -\cos\varphi & 1 \end{pmatrix}$$

5.5 The Metric Tensor and Einstein Notation

For a general four-dimensional space-time system (with time as coordinate *ct*), the metric is a **4×4 tensor**. The general form is:

$$ds^2 = \sum_{\mu v} g_{\mu v}\, dx^{\mu}\, dx^{v}$$

In Einstein notation:

$$\boldsymbol{ds^2 = g_{\mu v} dx^{\mu} dx^{v}}$$

Here, μ and v range from 0 to 3, with coordinates $x^0 = ct,\; x^1 = x,\; x^2 = y,\; x^3 = z$. The **metric tensor** $g_{\mu v}$ contains all information about the curvature of space-time.

Example of a metric tensor in matrix form:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}$$

If the coordinate system is **orthogonal**, then all **cross terms** (where $\mu \neq \nu$) are zero: $g_{\mu\nu} = 0$ for $\mu \neq \nu$.

The value of ds^2 remains **unchanged** when the coordinate system is changed, provided that the corresponding metric is adjusted correctly. That is to say:

$$\boldsymbol{ds^2 = g_{\mu\nu}(x)dx^{\mu}dx^{\nu} = g_{\alpha\beta}(y)dy^{\alpha}dy^{\beta}}$$

Here you see that $g_{\mu\nu}$ is the "weighting factor" that determines how the infinitesimal displacements in the μ- and ν- directions contribute to the length.

- The **diagonal elements** $g_{\mu\mu}$ can be viewed as the "scaling factors" for the corresponding coordinate directions.
- The **off-diagonal elements** $g_{\mu\nu}$ $(\mu \neq \nu)$ indicate whether the coordinate directions are *skew* (i.e., not perpendicular). In a sense, they are related to direction cosines (projections of one axis onto another).

In summary

- A coordinate system is a tool for structuring space; distances can be calculated within it.
- In orthogonal systems, Pythagoras' theorem applies; in non-orthogonal systems, the cosine rule applies.
- Curved coordinate systems are necessary to describe gravitational fields in general relativity.
- The **metric** $\boldsymbol{g_{\mu\nu}}$ contains all information about distance determination and curvature of space or space-time.

5.6 Transformation between two coordinate systems

As mentioned earlier, in a curved coordinate system, a coordinate system with straight lines can be used "locally," in an infinitely small area. For a four-dimensional coordinate system, each new coordinate in the new x-system has a linear relationship with all old coordinates in the old y-system, according to the equation:

$$dx^0 = \frac{\partial x^0}{\partial y^0}dy^0 + \frac{\partial x^0}{\partial y^1}dy^1 + \frac{\partial x^0}{\partial y^2}dy^2 + \frac{\partial x^0}{\partial y^3}dy^3$$

The same applies to the other three coordinates, leading to the general formula:

$$\boldsymbol{dx^m = \frac{\partial x^m}{\partial y^r}dy^r}$$

The summation is performed over the repeated index *r*.

Here, the summation is implicitly performed over the index *r* according to **Einstein notation**. This means that for each value of *m*, the derivatives over all values of *r* (from 0 to 3) are added together. This formula describes how an infinitesimal change in the new coordinate system x^m is constructed from changes in the old systemy^r .

5.6.1 Detailed Explanation of the Metric Tensor

We start with a Cartesian coordinate system, which in this case is similar to the Minkowski equation (see section5.10.1and Appendix 9.1equation 11a) in special relativity:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

To make the notation more compact and general, we rename the differential terms:

$$cdt = dx^0, dx = dx^1, dy = dx^2, dz = dx^3$$

Here, all differential terms have the dimension of length (meters). In this notation, the space-time interval is written as:

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$$

The **metric tensor** $\eta_{\mu\nu}$ (the Minkowski metric) is in matrix form:

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

This tensor describes the distance structure of flat space-time. The distance between two events is therefore:

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$$

Now we consider an arbitrary coordinate system y^α , with coordinates y^0, y^1, y^2, y^3 . The relationship between the old and the new system is given by the chain rule:

$$dx^\mu = \frac{\partial x^\mu}{\partial y^0} dy^0 + \frac{\partial x^\mu}{\partial y^1} dy^1 + \frac{\partial x^\mu}{\partial y^2} dy^2 + \frac{\partial x^\mu}{\partial y^3} dy^3$$

Or in compact notation:

$$dx^\mu = \frac{\partial x^\mu}{\partial y^\alpha} dy^\alpha$$

And:

$$dx^\nu = \frac{\partial x^\nu}{\partial y^\beta} dy^\beta$$

Substitution into the Minkowski form:

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$$

yields:

$$ds^2 = \boldsymbol{\eta_{\mu\nu} \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta}} dy^\alpha dy^\beta$$

We now define a new metric tensor $g_{\alpha\beta}$ in the coordinate system y^α , as follows:

$$g_{\alpha\beta} = \eta_{\mu\nu} \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta}$$

So that:

$$ds^2 = g_{\alpha\beta}\, dy^\alpha\, dy^\beta$$

If we then switch to another arbitrary coordinate system x^μ , the inverse transformation applies:

$$ds^2 = g_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu}\, dx^\mu\, dx^\nu = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

This gives the general transformation formula for the metric tensor:

$$\boldsymbol{g_{\mu\nu}(x) = \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}}(\mathbf{y})$$

This formula describes how the components of the metric tensor transform under a general coordinate transformation. It is a fundamental result in general relativity and forms the basis for understanding curved space-time.

5.7 Transformation between Cartesian and Polar (infinitesimal) Coordinates

As an example, we will now perform the transformation from a Cartesian to a spherical (polar) coordinate system. We assume that the reader is familiar with the standard transformation between the two systems (see figure below):

$$x = r \sin\theta \cos\varphi \qquad y = r \sin\theta \sin\varphi \qquad z = r\cos\theta$$

Derivation of *dx*, *dy*, and *dz*:

We differentiate the above expressions to obtain the infinitesimal displacements

$$\overrightarrow{dx} = \sin\theta\cos\varphi\, \overrightarrow{dr} + r\cos\theta\cos\varphi\, \overrightarrow{d\theta} - r\sin\theta\sin\varphi\, \overrightarrow{d\varphi}$$

$$\overrightarrow{dy} = \sin\theta\sin\varphi\, \overrightarrow{dr} + r\cos\theta\sin\varphi\, \overrightarrow{d\theta} + r\sin\theta\cos\varphi\, \overrightarrow{d\varphi}$$

$$\overrightarrow{dz} = \cos\theta\, \overrightarrow{dr} - r\sin\theta\, \overrightarrow{d\theta}$$

These vectorial differentials describe the infinitesimal displacements in the x, y, and z directions in terms of dr, $d\theta$, and $d\varphi$.

Determination of the square of the differentials

To determine the magnitude of *dx, dy*, and *dz*, we take the inner product of each of them:

$$dx^2 = \overrightarrow{dx}\cdot\overrightarrow{dx}; \quad dy^2 = \overrightarrow{dy}\cdot\overrightarrow{dy}; \qquad dz^2 = \overrightarrow{dz}\cdot\overrightarrow{dz}$$

Since the coordinates r, θ and φ are perpendicular to each other, the cross products are zero, resulting in:

$$dx^2 = \sin^2\theta\cos^2\varphi\, dr^2 + r^2\cos^2\theta\cos^2\varphi\, d\theta^2 + r^2\sin^2\theta\sin^2\varphi\, d\varphi^2$$

$$dy^2 = \sin^2\theta\sin^2\varphi\, dr^2 + r^2\cos^2\theta\sin^2\varphi\, d\theta^2 + r^2\sin^2\theta\cos^2\varphi\, d\varphi^2$$

$$dz^2 = \cos^2\theta\, dr^2 + r^2 \sin^2\theta\, d\theta^2$$

Addition of $dx^2+dy^2+dz^2$

By adding the three expressions above, we get:

$$dx^2+dy^2+dz^2 = \sin^2\theta\cos^2\varphi\, dr^2 + \sin^2\theta\sin^2\varphi\, dr^2 + \cos^2\theta\, dr^2 + r^2\cos^2\theta\cos^2\varphi\, d\theta^2 \\ + r^2\cos^2\theta\sin^2\varphi\, d\theta^2 + r^2\sin^2\theta\, d\theta^2 + r^2\sin^2\theta\sin^2\varphi\, d\varphi^2 + r^2\sin^2\theta\cos^2\varphi\, d\varphi^2$$

Using the trigonometric identities:

$$\cos^2\varphi + \sin^2\varphi = 1, \quad \cos^2\theta + \sin^2\theta = 1$$

this is simplified to:

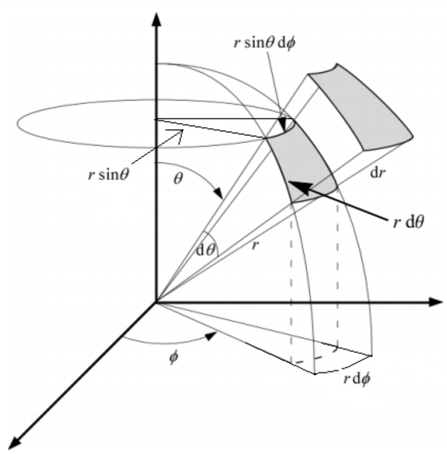
$$\boldsymbol{dx^2+dy^2+dz^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2} \tag{1}$$

This expression is precisely the spatial component of the metric in spherical coordinates. The time component can be added as $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$, or in spherical form:

$$ds^2 = c^2dt^2 - dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\varphi^2$$

Volume element in spherical coordinates

This describes the transformation from a system with Cartesian coordinates to a system with spherical (polar) coordinates.



The volume element in Cartesian coordinates is:

$$dV = dxdydz$$

After transformation to spherical coordinates, this becomes:

$$dV = dr.r\, d\theta.\mathrm{r}\sin\theta\, d\varphi = r^2\sin\theta\, drd\theta d\varphi$$

Calculation of the Volume of a Sphere

The total volume of a sphere with radius *R* follows from the integral:

$$V = \iiint r^2\sin\theta\, drd\theta d\varphi$$

with the integration limits:

- r∈ [0, R]
- θ∈ [0, π]
- φ∈ [0, 2π]

$$V = \int_0^R r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi$$

$$V = \frac{1}{3} r^3 |_0^R \cdot (-\cos\theta) |_0^\pi \cdot \varphi|_0^{2\pi} = \frac{1}{3} R^3 \cdot 2 \cdot 2\pi = \frac{4}{3}\pi R^3$$

This confirms the well-known result for the volume of a sphere.

5.8 Exercise: Applying the Metric Transformation Formula

Here we show how the **metric transformation formula** is formally applied when converting from a Cartesian to a polar (spherical) coordinate system.

1. General formulas

We recall the following relationships:

1.1. Transformation of coordinates

$$dx^m = \frac{\partial x^m}{\partial y^r} dy^r$$

1.2. Line element in Cartesian coordinates

$$ds^2 = \eta_{mn}\, d\xi^m d\xi^n$$

1.3. Invariance of the line element under coordinate transformation

$$ds^2 = g_{mn}(x) dx^m dx^n = g_{pq}(y) dy^p dy^q$$

1.4. Transformation formula for the metric

$$\boldsymbol{g_{pq}(y) = g_{mn}(x) \frac{dx^m}{dy^p} \frac{dx^n}{dy^q}}$$

2. From Cartesian to spherical

We consider the following Cartesian metric (in four-dimensional space-time with signature (+ ,– ,– ,–)):

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

The corresponding form in spherical coordinates is (with equation_4_6_1):

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\varphi^2$$

The metric tensor in Cartesian coordinates is:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

So here we **have** $g_{00} = 1$, $g_{11} = -1$, $g_{22} = -1$ en $g_{33} = -1$ for Cartesian elements, and the other elements are zero.

Our objective is now: **find the metric $\boldsymbol{g_{\mu\nu}}$ in spherical coordinates**.

$$g_{00} = 1\,, \; g_{11} = -1\,, \; g_{22} = -r^2, g_{33} = -r^2 \sin^2 \theta$$

3. Coordinate transformation

The spherical coordinates are expressed as a function of the Cartesian coordinates:

$$x = r \sin\theta \cos\varphi$$

$$y = r \sin\theta \sin\varphi$$

$$z = r \cos\theta$$

We apply the transformation formula:

$$dx^m = \frac{\partial x^m}{\partial y^r} dy^r$$

When we write out this formula in full, it looks like this:

$$dt = \frac{\partial t}{\partial t} dt + \frac{\partial t}{\partial r} dr + \frac{\partial t}{\partial \theta} d\theta + \frac{\partial t}{\partial \varphi} d\varphi$$

$$dx = \frac{\partial x}{\partial t} dt + \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \varphi} d\varphi$$

$$dy = \frac{\partial y}{\partial t} dt + \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \varphi} d\varphi$$

$$dz = \frac{\partial z}{\partial t} dt + \frac{\partial z}{\partial r} dr + \frac{\partial z}{\partial \theta} d\theta + \frac{\partial z}{\partial \varphi} d\varphi$$

The differentials become:

$$dt = dt$$

$$dx = \sin\theta \cos\varphi \, dr + r \cos\theta \cos\varphi \, d\theta - r \sin\theta \sin\varphi \, d\varphi$$

$$dy = \sin\theta \sin\varphi \, dr + r \cos\theta \sin\varphi \, d\theta + r \sin\theta \cos\varphi \, d\varphi$$

$$dz = \cos\theta \, dr - r \sin\theta \, d\theta$$

So the elements of the metric tensor are:

$$
\begin{array}{llll}
\frac{\partial t}{\partial t} = 1 & \frac{\partial t}{\partial r} = 0 & \frac{\partial t}{\partial \theta} = 0 & \frac{\partial t}{\partial \varphi} = 0 \\
\frac{\partial x}{\partial t} = 0 & \frac{\partial x}{\partial r} = +\sin\theta\cos\varphi & \frac{\partial x}{\partial \theta} = +r\cos\theta\cos\varphi & \frac{\partial x}{\partial \varphi} = -r\sin\theta\sin\varphi \\
\frac{\partial y}{\partial t} = 0 & \frac{\partial y}{\partial r} = +\sin\theta\sin\varphi & \frac{\partial y}{\partial \theta} = +r\cos\theta\sin\varphi & \frac{\partial y}{\partial \varphi} = +r\sin\theta\cos\varphi \\
\frac{\partial z}{\partial t} = 0 & \frac{\partial z}{\partial r} = +\cos\theta & \frac{\partial z}{\partial \theta} = -r\sin\theta & \frac{\partial z}{\partial \varphi} = 0
\end{array}
$$

4. Calculating the New Metric

We now apply the transformation formula:

$$g_{pq}(y) = g_{mn}(x)\frac{dx^m}{dy^p}\frac{dx^n}{dy^q}$$

We now work out the metric tensor elements:

$$
\begin{aligned}
g_{00}(y) = {} & g_{00}(x)\frac{dx^0}{dy^0}\frac{dx^0}{dy^0} + g_{01}(x)\frac{dx^0}{dy^0}\frac{dx^1}{dy^0} + g_{02}(x)\frac{dx^0}{dy^0}\frac{dx^2}{dy^0} + g_{03}(x)\frac{dx^0}{dy^0}\frac{dx^3}{dy^0} + \\
& g_{10}(x)\frac{dx^1}{dy^0}\frac{dx^0}{dy^0} + g_{11}(x)\frac{dx^1}{dy^0}\frac{dx^1}{dy^0} + g_{12}(x)\frac{dx^1}{dy^0}\frac{dx^2}{dy^0} + g_{13}(x)\frac{dx^1}{dy^0}\frac{dx^3}{dy^0} + \\
& g_{20}(x)\frac{dx^2}{dy^0}\frac{dx^0}{dy^0} + g_{21}(x)\frac{dx^2}{dy^0}\frac{dx^1}{dy^0} + g_{22}(x)\frac{dx^2}{dy^0}\frac{dx^2}{dy^0} + g_{23}(x)\frac{dx^2}{dy^0}\frac{dx^3}{dy^0} + \\
& g_{30}(x)\frac{dx^3}{dy^0}\frac{dx^0}{dy^0} + g_{31}(x)\frac{dx^3}{dy^0}\frac{dx^1}{dy^0} + g_{32}(x)\frac{dx^3}{dy^0}\frac{dx^2}{dy^0} + g_{33}(x)\frac{dx^3}{dy^0}\frac{dx^3}{dy^0}
\end{aligned}
$$

As an example, we now fill in the correct polar and Cartesian coordinates:

$$
\begin{aligned}
g_{rr} = {} & g_{tt}\frac{dt}{dr}\frac{dt}{dr} + g_{tx}\frac{dt}{dr}\frac{dx}{dr} + g_{ty}\frac{dt}{dr}\frac{dy}{dr} + g_{tz}\frac{dt}{dr}\frac{dz}{dr} + \\
& g_{xt}\frac{dx}{dr}\frac{dt}{dr} + g_{xx}\frac{dx}{dr}\frac{dx}{dr} + g_{xy}\frac{dx}{dr}\frac{dy}{dr} + g_{xz}\frac{dx}{dr}\frac{dz}{dr} + \\
& g_{yt}\frac{dy}{dr}\frac{dt}{dr} + g_{yx}\frac{dy}{dr}\frac{dx}{dr} + g_{yy}\frac{dy}{dr}\frac{dy}{dr} + g_{yz}\frac{dy}{dr}\frac{dz}{dr} + \\
& g_{zt}\frac{dz}{dr}\frac{dt}{dr} + g_{zx}\frac{dz}{dr}\frac{dx}{dr} + g_{zy}\frac{dz}{dr}\frac{dy}{dr} + g_{zz}\frac{dz}{dr}\frac{dz}{dr}
\end{aligned}
$$

Because the coordinate system here is an orthogonal system, only those elements with equal indices are non-zero.
So the matrix above ultimately becomes:

1.5. Time component:

$$g_{tt} = g_{tt}\frac{dt}{dt}\frac{dt}{dt} + g_{xx}\frac{dx}{dt}\frac{dx}{dt} + g_{yy}\frac{dy}{dt}\frac{dy}{dt} + g_{zz}\frac{dz}{dt}\frac{dz}{dt}$$

$$g_{tt} = 1 + 0 + 0 + 0 = 1$$

1.6. Radial component:

$$g_{rr} = g_{tt}\frac{dt}{dr}\frac{dt}{dr} + g_{xx}\frac{dx}{dr}\frac{dx}{dr} + g_{yy}\frac{dy}{dr}\frac{dy}{dr} + g_{zz}\frac{dz}{dr}\frac{dz}{dr}$$

$$g_{rr} = 0 - 1(+\sin\theta\cos\varphi)^2 - 1(+\sin\theta\sin\varphi)^2 - 1(+\cos\theta)^2 = -\sin^2\varphi - \cos^2\varphi = -1$$

1.7. Angular component $\boldsymbol{\theta}$:

$$g_{\theta\theta} = g_{tt}\frac{dt}{d\theta}\frac{dt}{d\theta} + g_{xx}\frac{dx}{d\theta}\frac{dx}{d\theta} + g_{yy}\frac{dy}{d\theta}\frac{dy}{d\theta} + g_{zz}\frac{dz}{d\theta}\frac{dz}{d\theta}$$

$$g_{\theta\theta} = 0 - 1(+r\cos\theta\cos\varphi)^2 - 1(+r\cos\theta\sin\varphi)^2 - 1(-r\sin\theta)^2 = -r^2\cos^2\theta - r^2\sin^2\theta = -r^2$$

1.8. Angular component φ :

$$g_{\varphi\varphi} = g_{tt}\frac{dt}{d\varphi}\frac{dt}{d\varphi} + g_{xx}\frac{dx}{d\varphi}\frac{dx}{d\varphi} + g_{yy}\frac{dy}{d\varphi}\frac{dy}{d\varphi} + g_{zz}\frac{dz}{d\varphi}\frac{dz}{d\varphi}$$

$$g_{\varphi\varphi} = 0 - 1(-r\sin\theta\sin\varphi)^2 - 1(+r\sin\theta\cos\varphi)^2 - 0 = -r^2\sin^2\varphi$$

Result

So the transformation from a Cartesian to a polar metric tensor yields the following elements:

$$\boldsymbol{g_{00} = 1 \quad g_{11} = -1 \quad g_{22} = -1 \qquad g_{33} = -1}$$
$$\boldsymbol{g_{tt} = 1 \quad g_{rr} = -1 \quad g_{\theta\theta} = -r^2 \quad g_{\varphi\varphi} = -r^2\sin^2\varphi}$$

Or in matrix form:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 sin^2\theta \end{pmatrix}$$

This corresponds to the metric of a polar coordinate system in a three-dimensional space.

Conclusion

Applying the metric transformation formula to the transition from Cartesian to spherical coordinates leads to the expected spherical form of the space-time metric. This exercise illustrates how tensor transformations are used to guarantee coordinate invariance of physical laws within general relativity.

5.9 Further Considerations on Co- and Contravariant Transformations

5.9.1 Introduction

In this section, we examine how basis vectors and vector components transform under a coordinate transformation. We consider both the direct and inverse transformations and check their consistency. These considerations form the basis for understanding covariant and contravariant objects in tensor analysis.

5.9.2 Covariant Transformation of Basis Vectors and Dual Vectors (or one-forms):

Consider a two-dimensional vector space with original basis vectors $\vec{e}_1$ and $\vec{e}_2$, which are transformed to a new coordinate system with basis vectors $\vec{e}_1^{\,'}$ and $\vec{e}_2^{\,'}$. This transformation is linear and can be written as:

$$\vec{e}_1^{\,'} = a_{11}\vec{e}_1 + a_{12}\vec{e}_2$$
$$\vec{e}_2^{\,'} = a_{21}\vec{e}_1 + a_{22}\vec{e}_2$$

This can be written in matrix form as:

$$\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

Or more concisely:

$$\vec{e}^{\,'} = A\vec{e}$$

5.9.2.1 Inverse Transformation of the Basis Vectors

To find the inverse transformation (from the transformed to the original system), we solve $\vec{e}_1$ and $\vec{e}_2$ in terms of $\vec{e}_1^{\,'}$ and $\vec{e}_2^{\,'}$.

Step 1: Set up linear combination

We take combinations of the original transformations to isolate $\vec{e}_1$:

$$a_{22}\vec{e}_1^{\,'} = a_{11}a_{22}\vec{e}_1 + a_{12}a_{22}\vec{e}_2$$
$$a_{12}\vec{e}_2^{\,'} = a_{12}a_{21}\vec{e}_1 + a_{12}a_{22}\vec{e}_2$$

Subtracting gives:

$$a_{22}\vec{e}_1^{\,'} - a_{12}\vec{e}_2^{\,'} = (a_{11}a_{22} - a_{12}a_{21})\vec{e}_1$$

This gives us the expression for $\vec{e}_1$:

$$\vec{e}_1 = \frac{\boldsymbol{a_{22}\vec{e}_1^{\,'} - a_{12}\vec{e}_2^{\,'}}}{\boldsymbol{a_{11}a_{22} - a_{12}a_{21}}}$$

Step 2: The same for $\vec{e}_2$

In a similar way, we find:

$$a_{21}\vec{e}_1^{\,'} = a_{11}a_{21}\vec{e}_1 + a_{12}a_{21}\vec{e}_2$$
$$a_{11}\vec{e}_2^{\,'} = a_{11}a_{21}\vec{e}_1 + a_{11}a_{22}\vec{e}_2$$

Now we multiply the first equation by a_{21} and the second by a_{11} and subtract them from each other again to find $\vec{e}_2$:

$$a_{21}\vec{e}_1^{\,'} - a_{11}\vec{e}_2^{\,'} = (a_{12}a_{21} - a_{11}a_{22})\vec{e}_2$$

So the expression for$\vec{e}_2$ is:

$$\vec{e}_2 = \frac{\boldsymbol{-a_{21}\vec{e}_1^{\,'} + a_{11}\vec{e}_2^{\,'}}}{\boldsymbol{a_{11}a_{22} - a_{12}a_{21}}}$$

In matrix form, the inverse transformation is:

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \frac{\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} e_1^{'} \\ e_2^{'} \end{pmatrix}$$

Or, more concisely:

$$\vec{e} = (A^{-1})\vec{e}^{\,'}$$

5.9.2.2 Checking the Inverse Transformation

We can check whether A and A^{-1} are indeed inverses of each other by multiplying the two matrices together and seeing whether they produce the identity matrix $A^{-1}A = I$:

$$\begin{pmatrix} e_1^{'} \\ e_2^{'} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

Now we multiply A^{-1} by A:

$$\frac{1}{(a_{11}a_{22} - a_{12}a_{21})} \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} =$$

$$\frac{\begin{pmatrix} a_{22}a_{11} - a_{21}a_{12} & a_{22}a_{21} - a_{21}a_{22} \\ -a_{12}a_{11} + a_{11}a_{12} & -a_{12}a_{21} + a_{11}a_{22} \end{pmatrix}}{(a_{11}a_{22} - a_{12}a_{21})} =$$

$$\frac{\begin{pmatrix} a_{22}a_{11} - a_{21}a_{12} & 0 \\ 0 & -a_{12}a_{21} + a_{11}a_{22} \end{pmatrix}}{(a_{11}a_{22} - a_{12}a_{21})} =$$

$$\frac{(a_{22}a_{11} - a_{21}a_{12}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}{(a_{11}a_{22} - a_{12}a_{21})} =$$

This simplifies to the identity matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore:

$$A^{-1}A = I$$

the inverse transformation is correct *Q.E.D*.

5.9.2.3 Conclusion

We have derived the covariant transformation for basis vectors and its inverse in a two-dimensional space. We have verified that the transformation and its inverse cancel each other out to the identity matrix, which confirms the consistency of the transformation between basis vectors in different coordinate systems. This formal consistency is essential for the correct application of tensor transformations in general relativity.

5.9.3 Contravariant Transformation of Vector Components

In differential geometry, it is essential to distinguish between how basis vectors (covariant) and how the components of vectors (contravariant) transform under a coordinate transformation. In this section, we examine the transformation properties of contravariant vector components in a two-dimensional space.

Vector Invariance and Component Transformation

A vector $\vec{V}$ remains geometrically the same under a coordinate transformation, but its components change. Suppose we write in the original coordinate system:

$$\vec{V} = V_1\vec{e}_1 + V_2\vec{e}_2$$

and in the new (transformed) system:

$$\vec{V} = V_1'\vec{e}_1' + V_2'\vec{e}_2'$$

V_i Since the vector itself remains invariant, the components of the vector must change as the basis vectors change.

Basis change

The new basis vectors are linearly related to the original basis vectors via a matrix *A*:

$$\begin{pmatrix} \vec{e}_1' \\ \vec{e}_2' \end{pmatrix} = \boldsymbol{A}\begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix}, \text{ where } \boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The inverse transformation for the basis vectors is then:

$$\begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix} = \boldsymbol{A}^{-1}\begin{pmatrix} \vec{e}_1' \\ \vec{e}_2' \end{pmatrix}$$

Derivation of the Contravariante Transformation

By expressing the basis vectors $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$ in terms of $\vec{e}_1'$ and $\vec{e}_2'$, we obtain:

$$\vec{V} = V_1\vec{e}_1 + V_2\vec{e}_2$$

$$\vec{V} = V_1\left(\frac{a_{22}\vec{e}_1^{'} - a_{12}\vec{e}_2^{'}}{a_{11}a_{22} - a_{12}a_{21}}\right) + V_2\left(\frac{-a_{21}\vec{e}_1^{'} + a_{11}\vec{e}_2^{'}}{a_{11}a_{22} - a_{12}a_{21}}\right)$$

This can be rewritten as:

$$\vec{V} = \left(\frac{a_{22}V_1\vec{e}_1^{'} - a_{12}V_1\vec{e}_2^{'}}{a_{11}a_{22} - a_{12}a_{21}}\right) + \left(\frac{-a_{21}V_2\vec{e}_1^{'} + a_{11}V_2\vec{e}_2^{'}}{a_{11}a_{22} - a_{12}a_{21}}\right)$$

By combining the terms:

$$\vec{V} = \frac{(a_{22}V_1 - a_{21}V_2)\vec{e}_1^{'} + (-a_{12}V_1 + a_{11}V_2)\vec{e}_2^{'}}{a_{11}a_{22} - a_{12}a_{21}}$$

And we also know that:

$$\vec{V} = V_1^{'}\vec{e}_1^{'} + V_2^{'}\vec{e}_2^{'}$$

This gives us the transformations for the components of the vector:

$$V_1^{'} = \frac{(a_{22}V_1 - a_{21}V_2)}{a_{11}a_{22} - a_{12}a_{21}}$$

$$V_2^{'} = \frac{(-a_{12}V_1 + a_{11}V_2)}{a_{11}a_{22} - a_{12}a_{21}}$$

In matrix form, we can write this as:

$$\begin{pmatrix} \boldsymbol{V_1^{'}} \\ \boldsymbol{V_2^{'}} \end{pmatrix} = \frac{\begin{pmatrix} \boldsymbol{a_{22}} & \boldsymbol{-a_{21}} \\ \boldsymbol{-a_{12}} & \boldsymbol{a_{11}} \end{pmatrix}}{\boldsymbol{a_{11}a_{22} - a_{12}a_{21}}} \begin{pmatrix} \boldsymbol{V_1} \\ \boldsymbol{V_2} \end{pmatrix}$$

So**:**

$$\vec{V}^{'} = (A^{-1})^T\vec{V}$$

Inverse Transformation of the Components

Now for the inverse transformation, starting with the transformed vector components:

$$\vec{V} = V_1^{'}\vec{e}_1^{'} + V_2^{'}\vec{e}_2^{'}$$

$$\vec{V} = V_1^{'}(a_{11}\vec{e}_1 + a_{12}\vec{e}_2) + V_2^{'}(a_{21}\vec{e}_1 + a_{22}\vec{e}_2)$$

This gives:

$$\vec{V} = (a_{11}V_1^{'} + a_{21}V_2^{'})\vec{e}_1 + (a_{12}V_1^{'} + a_{22}V_2^{'})\vec{e}_2$$

Which corresponds to:

$$\vec{V} = V_1\vec{e}_1 + V_2\vec{e}_2$$

From this, the relationships for the original vector components follow:

$$V_1 = a_{11}V_1^{'} + a_{21}V_2^{'}$$

$$V_2 = a_{12}V_1^{'} + a_{22}V_2^{'}$$

In matrix form:

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} V_1^{'} \\ V_2^{'} \end{pmatrix}$$

The basis vectors transform according to:

$$\begin{pmatrix} e_1^{'} \\ e_2^{'} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

In contrast, the vector components transform in the opposite direction:

$$\begin{pmatrix} V_1^{'} \\ V_2^{'} \end{pmatrix} = \frac{\begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix}}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

5.9.4 Summary of the Transformations

The core relationships are as follows:

Basis vectors (covariant):

$$\vec{e}^{\,'} = A\vec{e}$$

$$\vec{e} = (A^{-1})\vec{e}^{\,'}$$

Vector components (contravariant):

$$\vec{V}^{'} = (A^{-1})^T \vec{V}$$

$$\vec{V} = A^T \vec{V}^{'}$$

So if the basis vectors (covariant vectors) transform with:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \boldsymbol{A}$$

While the relationship between:

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \text{ en } \begin{pmatrix} V_1^{'} \\ V_2^{'} \end{pmatrix}$$

is

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \boldsymbol{A^T}$$

The transposed of *A:*

Then the contravariant vectors transform according to:

$$\begin{pmatrix} V_1^{'} \\ V_2^{'} \end{pmatrix} \text{ en } \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

$$\frac{\begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix}}{(a_{11}a_{22} - a_{12}a_{21})} = \left(\boldsymbol{A^{-1}}\right)^{\boldsymbol{T}}$$

This is the inverse and its transpose.

Conclusion

When the coordinate system changes, the basis vectors transform according to a matrix $\boldsymbol{A}$, while the vector components transform with the inverse transpose $\left(\boldsymbol{A}^{-1}\right)^{\boldsymbol{T}}$. This contravariant transformation ensures that the vector $\vec{V}$ itself remains invariant—its representation adapts to the changing basis so that the geometric meaning is preserved.

5.10 Considerations on the Minkowski and Schwarzschild Formulas

5.10.1 Minkowsk i -space

The Minkowski metric is used in special relativity, in which the effects of gravity and acceleration are neglected. In this context, reference frames move uniformly at constant speed relative to each other, and the coordinate system used is linear and flat.

Consider a point *K* in space-time with its own coordinate system. In this system, *K* is always located at the origin, so that only time progresses. The distance in space-time—the interval—is then given by:

$$s = c\tau,$$

where τ is the proper time, measured by a clock moving with *K*.

An observer is located elsewhere in space-time with a different inertial system, which moves relative to *K*. If the observer observes that *K* is moving in space, then the measured velocity of *K* is:

$$v^2 = \frac{(x^2 + y^2 + z^2)}{t^2}$$

The Minkowski metric in four-dimensional space-time is then written as:

$$s^2 = c^2t^2 - x^2 - y^2 - z^2$$

For an infinitesimal segment along a world line, the following applies:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

This differential segment is, as it were, a tangent to the world line in space-time. Even if the world line is curved (as in the case of acceleration or in the presence of gravity), we can locally approximate it as being composed of linear segments.

The coordinates *t*, *x*, *y*, and *z* represent four components of a space-time vector. In an orthogonal coordinate system (such as in Minkowski space), the interval can be calculated using a generalized Pythagorean theorem. If we take the time component as imaginary *ict,* and the space components as real, then the familiar Minkowski form follows.

To describe the general structure of an interval *s* in four dimensions, we can write:

We must realize that *t*, *x*, *y*, and *z* have magnitude and direction; they are vectors. Finding the magnitude of ***s*** means adding the four vectors. If this coordinate system is an orthogonal system, the Pythagorean theorem can be applied to the space segment. If we consider the time part as complex *icdt,* and for the left side of the formula *ds=icdτ,* then by squaring the coordinates we obtain the Minkowski formula.

$$\vec{s} = a_1\vec{x}_1 + a_2\vec{x}_2$$

To find the magnitude of *s*, we calculate the inner product of *s* by multiplying *s* by itself:

$$\vec{s}.\vec{s} = (a_1\vec{x}_1 + a_2\vec{x}_2).(a_1\vec{x}_1 + a_2\vec{x}_2),$$

Which results in:

$$s^2 = {a_1}^2 x_{1+}^2 a_1 a_2 \vec{x}_1.\vec{x}_2 + a_1 a_2 \vec{x}_2.\vec{x}_{1+} {a_2}^2 x_2^2$$

In four dimensions, we generalize this using the metric tensor$g_{\mu\nu}$:

$$s^2 = \sum_{\mu}\sum_{\nu} g_{\mu\nu}\, x^{\mu} x^{\nu}$$

Or in Einstein notation (sum over repeated low and high indices):

$$s^2 = g_{\mu\nu}\, x^{\mu} x^{\nu}$$

When using a local orthogonal coordinate system, all products involving $\mu \neq \nu$ disappear. If only an infinitesimally small local "area" is considered, *dx* is used instead of *x*, and the same applies to the other coordinates. Finally, the equation results in a Minkowski or Schwarzschild form:

$$ds^2 = (cdx^0)^2 - \left(dx^1\right)^2 - (dx^2)^2 - (dx^3)^2$$

General:

$$ds^2 = g_{00}(cdx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2$$

In Minkowski space, the components of the metric tensor are constant:

$$g_{00} = 1 \text{ and } g_{11} = g_{22} = g_{33} = -1$$

Meaning of the Minkowski formula

The Minkowski interval formula is:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2dt'^2 - dx'^2 - dy'^2 - dz'^2$$

The left side represents an object that is in its own (moving) reference frame: it experiences only a progression in its own time τ. An observer in the coordinate system *t*, *x*, *y*, *z* sees the object moving at a speed:

$$v^2 = \frac{(dx^2 + dy^2 + dz^2)}{dt^2}$$

And a second observer in another inertial frame t', x', y', z' measures:

$$v'^2 = \frac{(dx'^2 + dy'^2 + dz'^2)}{dt'^2}$$

The relationship between proper time τ and coordinate time t is given by:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2d\tau^2$$

So:

$$c^2d\tau^2 = c^2dt^2\left(1 - \frac{v^2}{c^2}\right) = \frac{c^2dt^2}{\gamma^2}$$

Here,τ is the so-called proper time, that is, the time of a moving clock located at the origin of its own moving coordinate system.

The relationship between the proper time τ in the *ds* system and the observer is:

$$d\tau^2 = \frac{dt^2}{\gamma^2}$$

$$dt = \gamma d\tau$$

where the Lorentz factorγ is defined as:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Because $\gamma \geq 1$, $d\tau \leq dt$: a moving clock runs slower than a clock at rest from the perspective of an external observer.

5.10.2 Transformations performed by Schwarzschild

The Schwarzschild metric is an extension of the Minkowski metric in that it also takes into account the effects of mass and gravity. In contrast to the flat space-time of special relativity, this results in a curved space-time. This curvature translates into a non-linear coordinate system, adapted to the spherical symmetry around a massive body.

From Cartesian to spherical

Schwarzschild starts with the usual flat (Cartesian) coordinates and performs a transformation to spherical coordinates $(r, \theta, \emptyset)$. This results in the following expression for the space-time interval (in natural units *G=c=1*, this is often written without *c*, but we retain it here for didactic clarity):

$$ds^2 = \sigma^2c^2dt^2 - \frac{dr^2}{\sigma^2} - r^2d\theta^2 - r^2\sin^2\theta \, d\emptyset^2$$

With:

$$\sigma^2 = 1 - \frac{2GM}{rc^2}$$

The determinant of the metric

The determinant *g* of the metric tensor appears in these coordinates:

$$g = \sigma^2.\left(\frac{-1}{\sigma^2}\right).(-r^2).(-r^2\sin^2\theta) = -r^4\sin^2\theta$$

However, Einstein wanted g=-1 to apply in suitable coordinates in his field equations (as in the Minkowski metric). Schwarzschild therefore investigates whether there is a coordinate transformation that satisfies this condition.

Transformation to new coordinates

To normalize the determinant to g=−1, Schwarzschild defines new coordinates (x_1, x_2, x_3) , based on:

$$\frac{dr}{dx_1} = \frac{1}{r^2}\,, \qquad \frac{d\theta}{dx_2} = \frac{1}{\sin\theta}, \qquad \frac{d\emptyset}{dx_3} = 1$$

Schwarzschild notes: "The new variables are the *polar coordinates with determinant 1*." To obtain these derivatives, he finds the following relations:

$$x_1 = \frac{r^3}{3}\,, \;\; x_2 = -\cos\theta\,, \;\; x_3 = \emptyset$$

These new coordinates transform the metric to:

$$ds^2 = \sigma^2c^2dt^2 - \frac{{dx_1}^2}{r^4\sigma^2} - r^2\cdot\frac{1}{\sin^2\theta}{dx_2}^2 - r^2\sin^2\theta\,{dx_3}^2$$

Substitution of trigonometric relations

Since $x_2 = -\cos\theta$, the following applies:

$${x_2}^2 = \cos^2\theta = 1 - \sin^2\theta => \sin^2\theta = 1 - {x_2}^2$$

We thus rewrite the metric as:

$$ds^2 = \sigma^2c^2dt^2 - \frac{{dx_1}^2}{r^4\sigma^2} - r^2\frac{1}{1 - {x_2}^2}{dx_2}^2 - r^2(1 - {x_2}^2){dx_3}^2$$

The new metric components

The components of the metric tensor $g_{\mu\nu}$ in these transformed coordinates are now:

$$\boldsymbol{g_{00}} = \sigma^2$$

$$\boldsymbol{g_{11}} = -\frac{1}{r^4\sigma^2}$$

$$\boldsymbol{g_{22}} = -\frac{r^2}{1 - {x_2}^2}$$

$$\boldsymbol{g_{33}} = -r^2(1 - {x_2}^2)$$

The determinant *g* of this tensor is now:

$$g = g_{00}\cdot g_{11}\cdot g_{22}\cdot g_{33} = -1$$

Exactly as desired. The transformation performed by Schwarzschild is therefore valid and results in a metric with determinant -1, despite the curved nature of space-time.

Special cases

In the specific case $\theta = 90^o$, the following applies $\cos\theta = 0$, and therefore $x_2 = 0$, which leads to:

$$\sin^2\theta = 1 \quad \Longrightarrow \quad ds^2 = \sigma^2 c^2 dt^2 - \frac{{dx_1}^2}{r^4\sigma^2} - r^2 {dx_2}^2 - r^2 {dx_3}^2$$

In this plane around the equatorial plane, the metric becomes even simpler in form.

5.11 Schwarzschild: "On the Gravitational Field of a Mass Point According to Einstein's Theory "

The aim of Karl Schwarzschild in his 1916 article was to find an exact solution to Einstein's field equations in a vacuum. This solution describes the space-time around a mass point moving along a geodesic line in a four-dimensional manifold, with the space-time interval *ds* at its center.

Conditions for the solution

The following conditions are imposed on the solution:

1. **Time independence**: All components of the metric are independent of the time coordinate x_4.
2. **No space-time coupling**: The mixed components $g_{\rho 4} = g_{4\rho} = 0$ for$\rho = 1, 2, 3$.
3. **Spherical symmetry**: The solution is invariant under orthogonal transformations (rotations) of $x_1, x_{2,}\, x_3$; this reflects spherical symmetry.
4. **Asymptotic flatness**: At infinite distance, the components of the metric tensor disappear, with the exception of the following limits:

$$\lim_{\mathrm{r}\to\infty} g_{44} = 1, \qquad \lim_{\mathrm{r}\to\infty} g_{11} = g_{22} = g_{33} = -1$$

From Cartesian to spherical coordinates

Schwarzschild starts with a general metric in Cartesian coordinates:

$$ds^2 = Fdt^2 - G(dx^2 + dy^2 + dz^2) - H(xdx + ydy + zdz)^2$$

He then performs the standard transformations to spherical coordinates:

$$x = r\sin\vartheta\cos\varphi, \quad y = r\sin\vartheta\sin\varphi, \quad z = r\cos\vartheta;$$

After substitution, we obtain the space-time interval in spherical coordinates:

$$ds^2 = Fdt^2 - G(dr^2 + r^2 d\vartheta^2 + r^2\sin\vartheta^2\, d\varphi^2) - Hr^2 dr^2$$

This is rewritten as:

$$ds^2 = Fdt^2 - (G + Hr^2)dr^2 - Gr^2(d\vartheta^2 + \sin\vartheta^2\, d\varphi^2)$$

Transformation to determinant 1

Because the determinant of the metric in this case is not equal to -1, Schwarzschild performs a transformation to new variables that satisfy this condition. He defines:

$$x_1 = \frac{r^3}{3}, \quad x_2 = -\cos\vartheta, \quad x_3 = \varphi$$

This makes the line segment:

$$ds^2 = Fdt^2 - \left(\frac{G}{r^4} + \frac{H}{r^2}\right)dx_1{}^2 - Gr^2\left[\frac{dx_2{}^2}{1 - x_2{}^2} + dx_3{}^2(1 - x_2{}^2)\right]$$

The Schwarzschild solution

By inserting this metric into the Einstein field equations and solving them in vacuum $\left(T_{\mu\nu} = 0\right)$, Schwarzschild finds the well-known solution:

$$ds^2 = \boldsymbol{c^2 d\tau^2} = \left(\mathbf{1} - \frac{\boldsymbol{2GM}}{\boldsymbol{c^2 r}}\right)\boldsymbol{c^2 dt^2} - \frac{\mathbf{1}}{\left(\mathbf{1} - \frac{\boldsymbol{2GM}}{\boldsymbol{c^2 r}}\right)}\boldsymbol{dr^2} - \boldsymbol{r^2 d\theta^2} - \boldsymbol{r^2}\,\mathbf{sin}^{\mathbf{2}}\,\boldsymbol{\theta^2}\,\boldsymbol{d\emptyset^2} \tag{12}$$

This equation describes the curved space-time around a spherically symmetric mass point in a vacuum. Although Schwarzschild began his derivation with Cartesian coordinates, the final solution is more convenient and insightful in spherical coordinates, given the spherical symmetry of the problem.

There is also a less common form of the Schwarzschild solution in Cartesian coordinates, which is as follows:

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 - (dx^2 + dy^2 + dz^2) - \frac{\frac{2GM}{c^2 r}}{\left(1 - \frac{2GM}{c^2 r}\right)r^2}(xdx + ydy + zdz)^2 \tag{13}$$

However, this form is rarely practical, because spherical coordinates are much better suited to the symmetry of the problem, for example in applications such as the description of black holes or the exterior of stars.

Sources

- **Schwarzschild, K.**, *On the Gravitational Field of a Point-Mass, According to Einstein's Theory*, published on January 13, 1916.
- **Oas, G.**, various discussions and analyses of the Schwarzschild solution. See also the *Bibliography* section at the end of this document.

The Schwarzschild solution is a cornerstone of general relativity and is frequently used in astrophysics in the study of black holes, neutron stars, and other objects with extremely strong gravitational fields.

Part VI – Validation of the Theory

6 Verification of whether the Schwarzschild Metric satisfies Einstein's Field Equations

We will now mathematically verify whether Schwarzschild's equation satisfies Einstein's field equations. We will do this first for the complete field equations and then for the simplified ones.

6.1 Verification using the full field equations

The general form of Einstein's field equations is as follows:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Here, the left side describes the geometry of space-time, while the right side represents the content of mass and energy. The constant λ is the cosmological constant, which is usually negligible in calculations on an astrophysical or planetary scale. Therefore, the simplified equation is usually used:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

The left side of the formula represents the geometry, while the right side is formed by the mass and energy. In vacuum regions—i.e., outside a mass—$T_{\mu\nu} = 0$, so the equation reduces to:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

The indices μ and ν each take four values (0 to 3), resulting in 16 coupled differential equations. The field equations are completely dependent on the metric tensor $g_{\mu\nu}$ and its first and second derivatives. This is because they are constructed exclusively from the Christoffel symbols and their derivatives, and the Christoffel symbols themselves are completely determined by the metric and its first derivatives.

6.1.1 The Schwarzschild solution

Karl Schwarzschild found an exact solution to the field equations in vacuum, assuming spherical symmetry. The metric is:

$$\boldsymbol{ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2}$$

with:

$$\sigma = \sqrt{1 - \frac{2GM}{c^2 r}}$$

The general form of the metric in coordinate notation is:

$$ds^2 = g_{00}dt^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\emptyset^2$$

This shows that only four of the sixteen metric components are non-zero. Therefore, there are only four relevant components of the Ricci tensor $R_{\mu\nu}$, namely:

$$R_{00}, \quad R_{11}, \quad R_{22}, \quad R_{33}$$

6.1.2 The Ricci tensor and Christoffel symbols

The Ricci tensor is defined as:

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu}$$

With the general expression:

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} = \Gamma^{\rho}_{\mu\nu,\rho} - \Gamma^{\rho}_{\rho\mu,\nu} + \Gamma^{\rho}_{\rho\lambda}\,\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\,\Gamma^{\lambda}_{\rho\mu}$$

Or written differently:

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} = \frac{\partial\Gamma^{\rho}_{\mu\nu}}{\partial x^{\rho}} - \frac{\partial\Gamma^{\rho}_{\rho\mu}}{\partial x^{\nu}} + \Gamma^{\rho}_{\rho\lambda}\,\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\,\Gamma^{\lambda}_{\rho\mu}$$

This formula is composed of derivatives and products of the Christoffel symbols. The general form of a Christoffel symbol is:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right\}$$

6.1.3 Simplification in vacuum

As indicated earlier, in vacuum:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$$

Here, *R* stands for the Ricci scalar and represents the curvature of the local space-time. The Ricci scalar is a measure of the total curvature of space-time at a given point and is calculated as the contraction of the Ricci tensor:

$$R = g^{\mu\nu}R_{\mu\nu}$$

This means that the Ricci scalar is a summary of how space-time curves in all directions, based on the information in the Ricci tensor. In the case of Einstein's field equations in a vacuum, *R*= 0, which means that the total space-time curvature is zero outside a massive source.

When the previous formula is multiplied by $g^{\mu\nu}$, we get:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$$

$$g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R = 0 \implies R - \frac{1}{2}(4)R = 0$$

This can only be true if *R = 0* and therefore $R_{\mu\nu} = 0$.

So, as a result of the relationship between R and $R_{\mu\nu}$, it is clear that:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

so the vacuum equation becomes:

$$R_{\mu\nu} = 0$$

6.1.4 Calculations and numerical verification

Based on the general form of the Ricci tensor and the Christoffel symbols, we have demonstrated both numerically (using a computer program) and theoretically that the Schwarzschild metric indeed satisfies the vacuum equation. $R_{\mu\nu} = 0$
The relevant expressions for the Ricci components in terms of Christoffel symbols are:

$$R_{00} = \boldsymbol{\Gamma^{1}_{00,1}} + \Gamma^{1}_{00}\,\Gamma^{1}_{11} + \Gamma^{1}_{00}\,\Gamma^{2}_{12} + \Gamma^{1}_{00}\,\Gamma^{3}_{13} - \Gamma^{0}_{01}\,\Gamma^{1}_{00}$$

$$R_{11} = -\boldsymbol{\Gamma^{0}_{10,1}} - \boldsymbol{\Gamma^{2}_{12,1}} - \boldsymbol{\Gamma^{3}_{13,1}} + \Gamma^{1}_{11}\Gamma^{0}_{10} + \Gamma^{1}_{11}\,\Gamma^{2}_{12} + \Gamma^{1}_{11}\,\Gamma^{3}_{13} - \Gamma^{0}_{10}\Gamma^{0}_{01} - \Gamma^{2}_{12}\,\Gamma^{2}_{21} - \Gamma^{3}_{13}\,\Gamma^{3}_{31}$$

$$R_{22} = \boldsymbol{\Gamma^{1}_{22,1}} - \boldsymbol{\Gamma^{3}_{23,2}} + \Gamma^{1}_{22}\,\Gamma^{0}_{10} + \Gamma^{1}_{22}\,\Gamma^{1}_{11} + \Gamma^{1}_{22}\,\Gamma^{3}_{13} - \Gamma^{2}_{21}\,\Gamma^{1}_{22} - \Gamma^{3}_{23}\,\Gamma^{3}_{32}$$

$$R_{33} = +\boldsymbol{\Gamma^{1}_{33,1}} + \boldsymbol{\Gamma^{2}_{33,2}} + \Gamma^{1}_{33}\,\Gamma^{0}_{10} + \Gamma^{1}_{33}\,\Gamma^{1}_{11} + \Gamma^{1}_{33}\,\Gamma^{2}_{12} - \Gamma^{3}_{31}\,\Gamma^{1}_{33} - \Gamma^{3}_{32}\,\Gamma^{2}_{33}$$

These equations were evaluated by deriving the Christoffel symbols from the Schwarzschild metric and substituting them into the above expressions. These symbols are summarized in Table 1 (see Appendix 1.2).

Note: In the literature, the Christoffel formula is sometimes given with a minus sign (–½) or a plus sign (+½) for the leading factor. In our approach, we have used a positive factor of ½. This convention proved to be consistent with the result that all relevant Ricci components are zero:

$$R_{00} = R_{11} = R_{22} = R_{33} = 0$$

which is required according to Einstein's field equations in vacuum. Therefore, the Christoffel formula is applied in the following form:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\}$$

6.1.5 Conclusion

Through both analytical and numerical evaluation of the Ricci tensor components, based on the Schwarzschild metric and associated Christoffel symbols, it has been demonstrated that this solution does indeed satisfy Einstein's field equations in vacuum. This makes the Schwarzschild solution an exact and physically consistent description of the space-time structure outside a spherically symmetric mass.

6.2 Checking R_{00}, R_{11}, R_{22} and R_{33} in the Schwarzschild metric

When checking the Einstein field equations in vacuum, the components of the Ricci tensor—
namely R_{00}, R_{11}, R_{22} and R_{33}— must be evaluated in the context of the Schwarzschild solution. This is done in spherical coordinates $(t, r, \theta, \emptyset)$.

The Schwarzschild metric is:

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta \, d\emptyset^2$$

With

$$\sigma^2 = 1 - \frac{R_s}{r}$$

Where $R_s = \frac{2GM}{c^2}$ is the Schwarzschild radius.

To determine the components of the Ricci tensor, we go through the following steps:

1. **Derive the Christoffel symbols** from the Schwarzschild metric in spherical coordinates;
2. **Substituting these symbols into the expressions for the Ricci tensor components**;
3. **Checking** that all relevant components $R_{\mu\nu}$ are equal to zero, in accordance with the Einstein field equations in vacuum.

The Christoffel symbols used and their derivatives can be found in Appendix 1.2.
The individual checks are listed below.

Check of R_{00}

The component R_{00} is given by:

$$\mathbf{R_{00}} = \Gamma^1_{00,1} + \Gamma^1_{00}\,\Gamma^1_{11} + \Gamma^1_{00}\,\Gamma^2_{12} + \Gamma^1_{00}\,\Gamma^3_{13} - \Gamma^0_{01}\,\Gamma^1_{00}$$

After substituting the relevant terms, the following applies:

$$R_{00} = \frac{R_s(3R_s - 2r)}{2r^4} + \frac{\sigma^2 R_s}{2r^2}\frac{-R_s}{2r^2\sigma^2} + \frac{\sigma^2 R_s}{2r^2}\frac{1}{r} + \frac{\sigma^2 R_s}{2r^2}\frac{1}{r} - \frac{R_s}{2r^2\sigma^2}\frac{\sigma^2 R_s}{2r^2}$$

$$R_{00} = \frac{R_s(3R_s - 2r)}{2r^4} - \frac{{R_s}^2}{2r^4} + \frac{2R_s(r - Rs)}{2r^4} = \frac{3{R_s}^2 - 2rRs - {R_s}^2 + 2R_s r - 2{R_s}^2}{2r^4} = 0$$

So:

$$R_{00} = 0 \quad \text{q.e.d.}$$

Checking R_{11}

The component R_{11} is calculated from:

$$\mathbf{R_{11}} = -\mathbf{\Gamma^0_{10,1}} - \mathbf{\Gamma^2_{12,1}} - \mathbf{\Gamma^3_{13,1}} + \Gamma^1_{11}\Gamma^0_{10} + \Gamma^1_{11}\,\Gamma^2_{12} + \Gamma^1_{11}\,\Gamma^3_{13} - \Gamma^0_{10}\Gamma^0_{01} - \Gamma^2_{12}\,\Gamma^2_{21} - \Gamma^3_{13}\,\Gamma^3_{31}$$

After calculation:

$$R_{11} = -\frac{R_s(R_s - 2r)}{2r^4\sigma^4} - \frac{-1}{r^2} - \frac{-1}{r^2} + \frac{-R_s}{2r^2\sigma^2}\frac{R_s}{2r^2\sigma^2} + \frac{-R_s}{2r^2\sigma^2}\frac{1}{r} + \frac{-R_s}{2r^2\sigma^2}\frac{1}{r} - \frac{R_s}{2r^2\sigma^2}\frac{R_s}{2r^2\sigma^2} - \frac{1}{r}\frac{1}{r} - \frac{1}{r}\frac{1}{r}$$

$$R_{11} = -\frac{R_s(R_s - 2r)}{2r^4\sigma^4} + \frac{1}{r^2} + \frac{1}{r^2} - \frac{{R_s}^2}{4r^4\sigma^4} - \frac{R_s}{2r^3\sigma^2} - \frac{R_s}{2r^3\sigma^2} - \frac{{R_s}^2}{4r^4\sigma^4} - \frac{1}{r^2} - \frac{1}{r^2}$$

$$R_{11} = -\frac{R_s(R_s - 2r)}{2r^4\sigma^4} - \frac{{R_s}^2}{2r^4\sigma^4} - \frac{2R_s r(1 - \frac{Rs}{r})}{2r^4\sigma^4} = -\frac{R_s(R_s - 2r)}{2r^4\sigma^4} - \frac{{R_s}^2}{2r^4\sigma^4} - \frac{2R_s r - 2{R_s}^2}{2r^4\sigma^4}$$

$$R_{11} = \frac{-{R_s}^2 + 2rRs}{2r^4\sigma^4} + \frac{-{R_s}^2}{2r^4\sigma^4} + \frac{-2R_s r + 2{R_s}^2}{2r^4\sigma^4} = 0$$

Therefore:

$$R_{11} = 0 \quad \text{q.e.d.}$$

Check of R_{22}

For the component R_{22} , the following applies:

$$\boldsymbol{R_{22}} = \boldsymbol{\Gamma^1_{22,1}} - \boldsymbol{\Gamma^3_{23,2}} + \Gamma^1_{22}\,\Gamma^0_{10} + \Gamma^1_{22}\,\Gamma^1_{11} + \Gamma^1_{22}\,\Gamma^3_{13} - \Gamma^2_{21}\Gamma^1_{22} - \Gamma^3_{23}\,\Gamma^3_{32}$$

Filling in gives:

$$R_{22} = -1 + 1 - r\sigma^2\ \frac{R_s}{2r^2\sigma^2} + r\sigma^2\ \frac{+R_s}{2r^2\sigma^2} - r\sigma^2\ \frac{1}{r} + \frac{1}{r}\ r\sigma^2 - \ 0 = 0$$

Therefore:

$$R_{22} = 0 \quad \text{q.e.d.}$$

Check of R_{33}

The component R_{33} is calculated as follows:

$$\boldsymbol{R_{33}} = +\boldsymbol{\Gamma^1_{33,1}} + \boldsymbol{\Gamma^2_{33,2}} + \Gamma^1_{33}\,\Gamma^0_{10} + \Gamma^1_{33}\,\Gamma^1_{11} + \Gamma^1_{33}\,\Gamma^2_{12} - \Gamma^3_{31}\,\Gamma^1_{33} - \Gamma^3_{32}\,\Gamma^2_{33}$$

The calculation results in:

$$R_{33} = -1 + 1 - r\sigma^2\ \frac{R_s}{2r^2\sigma^2} + r\sigma^2\ \frac{R_s}{2r^2\sigma^2} - r\sigma^2\ \frac{1}{r} + \frac{1}{r} r\sigma^2 - \ 0 = 0$$

Therefore:

$$R_{33} = 0 \quad \text{q.e.d.}$$

Conclusion

All relevant components of the Ricci tensor are zero:

$$R_{\mu\nu} = 0 \ \text{ for } \ \mu, \nu \ \in \ \{0, 1, 2, 3\},$$

which confirms that the Schwarzschild solution satisfies Einstein's field equations in vacuum. This demonstrates that this solution correctly describes space-time outside a spherically symmetric mass in the absence of energy or matter. This constitutes a fundamental confirmation of both the consistency of the Schwarzschild solution and the correctness of general relativity in this specific case.

6.3 Checking the Ricci tensor components R₀₀, R₁₁, R₂₂, and R₃₃in Schwarzschild coordinates

We explicitly check the components of the Ricci tensor for the Schwarzschild solution in a modified coordinate system $(t_\infty, x_1, x_2, x_3)$, where the metric has the form:

$$ds^2 = \sigma^2 c^2 dt_\infty{}^2 - \frac{dx_1{}^2}{r^4\sigma^2} - \frac{r^2 dx_2{}^2}{\sin^2\theta} - r^2 \sin^2\theta\, dx_3{}^2$$

Where $\sigma^2 = 1 - \frac{R_s}{r}$, with R_s the Schwarzschild radius.

The necessary Christoffel symbols and their derivatives are given in Appendix 1.3. Below, we verify that all components of the Ricci tensor are zero, as required for a vacuum solution ($R_{\mu\nu} = 0$).

Component R_{00}

$$\boldsymbol{R_{00}} = \Gamma^1_{00,1} + \Gamma^1_{00}\,\Gamma^1_{11} + \Gamma^1_{00}\,\Gamma^2_{12} + \Gamma^1_{00}\,\Gamma^3_{13} - \Gamma^0_{01}\,\Gamma^1_{00}$$

Filled in with expressions for the Christoffel symbols:

$$R_{00} = \frac{\mathrm{R_s}^2}{2\mathrm{r}^4} + \frac{R_s\sigma^2}{2}\frac{3R_s - 4r}{2r^4\sigma^2} + \frac{R_s\sigma^2}{2}\frac{1}{r^3} + \frac{R_s\sigma^2}{2}\frac{1}{r^3} - \frac{R_s}{2r^4\sigma^2}\frac{R_s\sigma^2}{2}$$

$$R_{00} = \frac{2\mathrm{R_s}^2}{4\mathrm{r}^4} + \frac{3\mathrm{R_s}^2 - 4rR_s}{4r^4} + \frac{4R_s r\sigma^2}{4r^4} - \frac{\mathrm{R_s}^2}{4r^4}$$

$$R_{00} = \frac{2\mathrm{R_s}^2 + 3\mathrm{R_s}^2 - 4rR_s - \mathrm{R_s}^2}{4r^4} + \frac{4R_s(r - R_s)}{4r^4} = \frac{2\mathrm{R_s}^2 + 3\mathrm{R_s}^2 - 4rR_s - \mathrm{R_s}^2}{4r^4} + \frac{4R_s r - 4\mathrm{R_s}^2}{4r^4}$$

$$R_{00} = \frac{4\mathrm{R_s}^2 - 4rR_s}{4r^4} + \frac{4R_s r - 4\mathrm{R_s}^2}{4r^4} = 0$$

Result:

$$R_{00} = 0 \quad \text{q.e.d.}$$

Component R_{11}

$$\boldsymbol{R_{11}} = -\boldsymbol{\Gamma^0_{10,1}} - \boldsymbol{\Gamma^2_{12,1}} - \boldsymbol{\Gamma^3_{13,1}} + \Gamma^1_{11}\Gamma^0_{10} + \Gamma^1_{11}\,\Gamma^2_{12} + \Gamma^1_{11}\,\Gamma^3_{13} - \Gamma^0_{10}\Gamma^0_{01} - \Gamma^2_{12}\,\Gamma^2_{21} - \Gamma^3_{13}\,\Gamma^3_{31}$$

Filling in gives:

$$R_{11} = -\frac{R_s(3R_s - 4r)}{2r^8\sigma^4} - \frac{-3}{r^6} - \frac{-3}{r^6} + \frac{3R_s - 4r}{2r^4\sigma^2}\frac{R_s}{2r^4\sigma^2} + \frac{3R_s - 4r}{2r^4\sigma^2}\frac{1}{r^3} + \frac{3R_s - 4r}{2r^4\sigma^2}\frac{1}{r^3} - \frac{R_s}{2r^4\sigma^2}\frac{R_s}{2r^4\sigma^2} - \frac{1}{r^3}\frac{1}{r^3} - \frac{1}{r^3}\frac{1}{r^3}$$

After simplification:

$$R_{11} = -\frac{2R_s(3R_s - 4r)}{4r^8\sigma^4} + \frac{4}{r^6} + \frac{R_s(3R_s - 4r)}{4r^8\sigma^4} + \frac{4(3R_s - 4r)r(1 - \frac{Rs}{r})}{4r^8\sigma^4} - \frac{{R_s}^2}{4r^8\sigma^4}$$

$$R_{11} = \frac{-6{R_s}^2 + 8rR_s + 3{R_s}^2 - 4rR_s + 12R_s r - 16r^2 - 12{R_s}^2 + 16rRs - {R_s}^2}{4r^8\sigma^4} + \frac{4}{r^6}$$

$$R_{11} = \frac{-16{R_s}^2 + 32rR_s - 16r^2}{4r^8\sigma^4} + \frac{4}{r^6} = \frac{-16{R_s}^2 + 32rR_s - 16r^2}{4r^8\sigma^4} + \frac{16r^2\left(1 - \frac{Rs}{r}\right)^2}{4r^8\sigma^4}$$

$$R_{11} = \frac{-16{R_s}^2 + 32rR_s - 16r^2}{4r^8\sigma^4} + \frac{4}{r^6} = \frac{-16{R_s}^2 + 32rR_s - 16r^2}{4r^8\sigma^4} + \frac{16r^2(1 - 2\frac{Rs}{r} + \frac{{R_s}^2}{r^2})}{4r^8\sigma^4} =$$

$$R_{11} = \frac{-16{R_s}^2 + 32rR_s - 16r^2}{4r^8\sigma^4} + \frac{4}{r^6} = \frac{-16{R_s}^2 + 32rR_s - 16r^2}{4r^8\sigma^4} + \frac{16r^2 - 32rRs + 16{R_s}^2}{4r^8\sigma^4} = 0$$

Result:

$$R_{11} = 0 \quad \text{q.e.d.}$$

Component R_{22}

$$\boldsymbol{R_{22}} = \boldsymbol{\Gamma^1_{22,1}} - \boldsymbol{\Gamma^3_{23,2}} + \Gamma^1_{22}\,\Gamma^0_{10} + \Gamma^1_{22}\,\Gamma^1_{11} + \Gamma^1_{22}\,\Gamma^3_{13} - \Gamma^2_{21}\,\Gamma^1_{22} - \Gamma^3_{23}\,\Gamma^3_{32}$$

Filling in and simplification gives:

$$R_{22} = -3 + \frac{2R_s}{r} + 1 - r^3\sigma^2\,\frac{R_s}{2r^4\sigma^2} - r^3\sigma^2\,\frac{3R_s - 4r}{2r^4\sigma^2} - r^3\sigma^2\,\frac{1}{r^3} + \frac{1}{r^3}\,r^3\sigma^2 - 0$$

$$R_{22} = -3 + \frac{2R_s}{r} + 1 - \frac{R_s}{2r} - \frac{3R_s - 4r}{2r} - r^3\sigma^2\,\frac{1}{r^3} + \frac{1}{r^3}\,r^3\sigma^2 - 0$$

$$R_{22} = \frac{-4r}{2r} + \frac{4R_s}{2r} - \frac{R_s}{2r} - \frac{3R_s - 4r}{2r} = 0$$

Result:

$$R_{22} = 0 \quad \text{q.e.d.}$$

Component R_{33}

$$R_{33} = +\boldsymbol{\Gamma^1_{33,1}} + \boldsymbol{\Gamma^2_{33,2}} + \Gamma^1_{33}\,\Gamma^0_{10} + \Gamma^1_{33}\,\Gamma^1_{11} + \Gamma^1_{33}\,\Gamma^2_{12} - \Gamma^3_{31}\,\Gamma^1_{33} - \Gamma^3_{32}\,\Gamma^2_{33}$$

$$R_{33} = -3 + \frac{2R_s}{r} + 1 - r^3\sigma^2\,\frac{R_s}{2r^4\sigma^2} - r^3\sigma^2\,\frac{3R_s - 4r}{2r^4\sigma^2} - r^3\sigma^2\,\frac{1}{r^3} + \frac{1}{r^3}\,r^3\sigma^2 - 0$$

$$R_{33} = -3 + \frac{2R_s}{r} + 1 - \frac{R_s}{2r} - \frac{3R_s - 4r}{2r} - r^3\sigma^2\,\frac{1}{r^3} + \frac{1}{r^3}\,r^3\sigma^2 - 0$$

$$R_{33} = \frac{-4r}{2r} + \frac{4R_s}{2r} - \frac{R_s}{2r} - \frac{3R_s - 4r}{2r} = 0$$

Result:

$$R_{33} = 0 \quad \text{q.e.d.}$$

Conclusion

The four independent components of the Ricci tensor are all zero in Schwarzschild coordinates, as expected for the vacuum solution of Einstein's field equations:

$$R_{00} = R_{11} = R_{22} = R_{33} = 0$$

This confirms that the Schwarzschild metric is indeed a solution of $R_{\mu\nu} = 0$ outside the central mass.

6.4 Verification of the Schwarzschild Solution using a simplified form of the field equations

In this chapter, we check the Schwarzschild solution using a simplified version of the Einstein field equations. This limited form comes from Schwarzschild's original derivation and applies only when the trace of the metric tensor satisfies:

$$\mathrm{tr}(g_{\mu\nu}) = -1$$

In this approach, the field equations take the form:

$$G_{\mu\nu} = \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x_{\alpha}} + \Gamma^{\alpha}_{\mu\beta}\ \Gamma^{\beta}_{\nu\alpha}$$

In this expression, the **Christoffel symbols** are defined with a negative sign, as Schwarzschild did:

$$\Gamma^{\rho}_{\mu\nu} = -\frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\}$$

Due to this negative convention, all derived expressions, including the derivatives of the Ricci tensor, will differ in sign from the standard definition.

Schwarzschild used the coordinates (*t, x, y, z*) in his derivation. We will therefore start with these coordinates and show the relevant components of the Ricci tensor, as they follow from the limited formula.

Derivative components of the Ricci tensor

The following expressions apply in Schwarzschild's notation:

- For the R_{00} component:

$$R_{00} = \boldsymbol{\Gamma^{1}_{00,1}} + \Gamma^{0}_{01}\ \Gamma^{1}_{00} + \Gamma^{1}_{00}\ \Gamma^{0}_{10}$$

- For the R_{11} component:

$$R_{11} = \boldsymbol{\Gamma^{1}_{11,1}} + \Gamma^{0}_{10}\Gamma^{0}_{01} + \Gamma^{1}_{11}\ \Gamma^{1}_{11} + \Gamma^{2}_{12}\ \Gamma^{2}_{21} + \Gamma^{3}_{13}\ \Gamma^{3}_{31}$$

- For the R_{22} component:

$$R_{22} = \boldsymbol{\Gamma^{1}_{22,1}} + \boldsymbol{\Gamma^{2}_{22,2}} + \Gamma^{1}_{22}\ \Gamma^{2}_{12} + \Gamma^{2}_{21}\ \Gamma^{1}_{22} + \Gamma^{2}_{22}\ \Gamma^{2}_{22} + \Gamma^{3}_{23}\ \Gamma^{3}_{23}$$

- For the R_{33} component:

$$R_{33} = +\boldsymbol{\Gamma^{1}_{33,1}} + \boldsymbol{\Gamma^{2}_{33,2}} + \Gamma^{1}_{33}\ \Gamma^{3}_{13} + \Gamma^{2}_{33}\ \Gamma^{3}_{23} + \Gamma^{3}_{31}\ \Gamma^{1}_{33} + \Gamma^{3}_{32}\ \Gamma^{2}_{33}$$

These components can be easily filled in once the correct Christoffel symbols have been calculated from the Schwarzschild metric. In the next section, we evaluate these components explicitly.

First:

6.5 t, x, y, z (-modified polar) coordinates

We work in modified Cartesian-polar coordinates *(t, x, y, z),* in which Schwarzschild originally formulated his solution. We explicitly check that the components of the Ricci tensor $R_{\mu\nu}$ are zero

Calculation of R_{00}

$$R_{00} = \mathbf{\Gamma^{1}_{00,1}} + \Gamma^{0}_{01}\,\Gamma^{1}_{00} + \Gamma^{1}_{00}\,\Gamma^{0}_{10}$$

With the given values:

$$R_{00} = \frac{-R_s{}^2}{2r^4} + \frac{R_s}{2r^4\sigma^2}\frac{R_s\sigma^2}{2} + \frac{R_s\sigma^2}{2}\frac{R_s}{2r^4\sigma^2} = \frac{-R_s{}^2}{2r^4} + \frac{R_s{}^2}{2r^4} = 0$$

Result:

$$R_{00} = 0, \mathrm{q.e.d.}$$

Calculation of R_{11}

$$R_{11} = \boldsymbol{\Gamma^{1}_{11,1}} + \Gamma^{0}_{10}\Gamma^{0}_{01} + \Gamma^{1}_{11}\,\Gamma^{1}_{11} + \Gamma^{2}_{12}\,\Gamma^{2}_{21} + \Gamma^{3}_{13}\,\Gamma^{3}_{31}$$

By carefully substituting and simplifying all terms, we find:

$$R_{11} = \frac{-6}{r^6\sigma^4} + \frac{10R_s}{r^7\sigma^4} - \frac{4.5R_s{}^2}{r^8\sigma^4} + \frac{R_s}{2r^4\sigma^2}\frac{R_s}{2r^4\sigma^2} + \frac{3R_s - 4r}{2r^4\sigma^2}\,\frac{3R_s - 4r}{2r^4\sigma^2} + \frac{1}{r^3}\,\frac{1}{r^3} + \frac{1}{r^3}\,\frac{1}{r^3}$$

$$R_{11} = \frac{-6}{r^6\sigma^4} + \frac{10R_s}{r^7\sigma^4} - \frac{4.5R_s{}^2}{r^8\sigma^4} + \frac{R_s{}^2}{4r^8\sigma^4} + \frac{9R_s{}^2 + 16r^2 - 24rR_s}{4r^8\sigma^4} + \frac{2}{r^6}$$

$$R_{11} = \frac{-24r^2}{4r^8\sigma^4} + \frac{40rR_s}{4r^8\sigma^4} - \frac{18R_s{}^2}{4r^8\sigma^4} + \frac{R_s{}^2}{4r^8\sigma^4} + \frac{9R_s{}^2 + 16r^2 - 24rR_s}{4r^8\sigma^4} + \frac{2}{r^6}$$

$$R_{11} = \frac{-8R_s{}^2 - 8r^2 + 16rR_s}{4r^8\sigma^4} + \frac{2}{r^6}$$

$$R_{11} = \frac{-8R_s{}^2 - 8r^2 + 16rR_s}{4r^8\sigma^4} + \frac{8r^2\sigma^4}{4r^8\sigma^4}$$

$$R_{11} = \frac{-8R_s{}^2 - 8r^2 + 16rR_s}{4r^8\sigma^4} + \frac{8r^2\left(1 - \frac{R_s}{r}\right)^2}{4r^8\sigma^4}$$

$$R_{11} = \frac{-8R_s{}^2 - 8r^2 + 16rR_s}{4r^8\sigma^4} + \frac{8\left(r^2 + R_s{}^2 - 2rR_s\right)}{4r^8\sigma^4} = 0$$

Result:

$$R_{11} = 0 \quad \mathrm{q.e.d.}$$

Calculation of R_{22}

$$R_{22} = \mathbf{\Gamma^{1}_{22,1}} + \mathbf{\Gamma^{2}_{22,2}} + \Gamma^{1}_{22}\,\Gamma^{2}_{12} + \Gamma^{2}_{21}\,\Gamma^{1}_{22} + \Gamma^{2}_{22}\,\Gamma^{2}_{22} + \Gamma^{3}_{23}\,\Gamma^{3}_{23}$$

After simplifying the trigonometric and radial terms:

$$R_{22} = \frac{-2R_s + 3r}{r\sin^2\theta} + \frac{-1 - \cos^2\theta}{\sin^4\theta} + \frac{-r^3\sigma^2}{\sin^2\theta}\,\frac{1}{r^3} + \frac{1}{r^3}\,\frac{-r^3\sigma^2}{\sin^2\theta} + \frac{-\cos(\theta)}{\sin^2(\theta)}\frac{-\cos(\theta)}{\sin^2(\theta)} + \frac{\cos\theta}{\sin^2(\theta)}\frac{\cos\theta}{\sin^2(\theta)}$$

$$R_{22} = \frac{-2R_s + 3r}{r\, sin^2\, \theta} + \frac{-1 - cos^2\, \theta}{sin^4\, \theta} + \frac{-2r^3 \sigma^2}{r^3\, sin^2\, \theta} + \frac{2cos^2\, \theta}{sin^4\, \theta}$$

$$R_{22} = \frac{-2R_s + 3r}{r\, sin^2\, \theta} + \frac{-1 - cos^2\, \theta}{sin^4\, \theta} + \frac{-2(r - R_s)}{r\, sin^2\, \theta} + \frac{2cos^2\, \theta}{sin^4\, \theta}$$

$$R_{22} = \frac{1}{sin^2\, \theta} + \frac{-1 - cos^2\, \theta}{sin^4\, \theta} + \frac{2\, cos^2\, \theta}{sin^4\, \theta}$$

$$R_{22} = \frac{sin^2\, \theta}{sin^4\, \theta} + \frac{-\, sin^2\, \theta\ - cos^2\, \theta - cos^2\, \theta}{sin^4\, \theta} + \frac{2cos^2\, \theta}{sin^4\, \theta} = 0$$

Result:

$$R_{22} = 0 \quad \text{q.e.d.}$$

Calculation of R_{33}

$$R_{33} = +\boldsymbol{\Gamma^{1}_{33,1}} + \boldsymbol{\Gamma^{2}_{33,2}} + \Gamma^{1}_{33}\, \Gamma^{3}_{13} + \Gamma^{2}_{33}\, \Gamma^{3}_{23} + \Gamma^{3}_{31}\, \Gamma^{1}_{33} + \Gamma^{3}_{32}\, \Gamma^{2}_{33}$$

After combining terms with the angle-dependent factors:

$$R_{33} = \left(3 - \frac{2R_s}{r}\right).\, sin^2\, \theta\ \ + 3\, cos^2\, \theta - 1 - r^3 \sigma^2\, sin^2\, \theta\, \frac{1}{r^3} + (-sin^2\, \theta\, cos\, \theta)\, \frac{cos\, \theta}{sin^2(\theta)}$$
$$- \frac{1}{r^3}\, r^3 \sigma^2\, sin^2\, \theta + \frac{cos\, \theta}{sin^2(\theta)} (-sin^2\, \theta\, cos\, \theta)$$

$$R_{33} = \left(3 - \frac{2R_s}{r}\right).\, sin^2\, \theta\ \ + 3\, cos^2\, \theta - 1 - 2\sigma^2\, sin^2\, \theta - \ 2sin^2(\theta)\, cos\, \theta \frac{cos\, \theta}{sin^2(\theta)}$$

$$R_{33} = \left(3 - \frac{2R_s}{r}\right).\, sin^2\, \theta\ \ + 3\, cos^2\, \theta - 1 - 2\left(1 - \frac{R_s}{r}\right).\, sin^2\, \theta - \ 2cos^2\, \theta$$

$$R_{33} = \left(3 - \frac{2R_s}{r}\right).\, sin^2\, \theta\ \ + 3\, cos^2\, \theta - 1 + \left(-2 + \frac{2R_s}{r}\right).\, sin^2\, \theta - \ 2cos^2\, \theta$$

$$R_{33} = sin^2\, \theta\ \ + 3\, cos^2\, \theta - 1 - \ 2cos^2\, \theta$$

$$R_{33} = sin^2\, \theta\ \ + 3\, cos^2\, \theta - sin^2\, \theta\ \ - cos^2\, \theta - \ 2cos^2\, \theta = 0$$

Result:

$$R_{33} = 0, \quad \text{q.e.d.}$$

Conclusion

We have shown that the components R_{00}, R_{11}, R_{22} en R_{33} of the Ricci tensor are all zero within Schwarzschild geometry when we use the simplified field equation withtr$(g_{\mu\nu}) = -1$. This confirms that the Schwarzschild solution is indeed a vacuum solution of the Einstein field equations, even in this specific derivation method.

6.6 Checking the Ricci Components in Spherical Coordinates

We check whether the Schwarzschild solution in spherical coordinates satisfies the *restricted* Einstein field equations, in which the determinant of the metric is g=−1.

The Schwarzschild metric in spherical coordinates is:

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta\, d\emptyset^2, \qquad \text{with } \sigma^2 = 1 - \frac{R_s}{r}$$

Below, we evaluate the components of the Ricci tensor $R_{\mu\nu}$ separately.

Component R_{00}

$$R_{00} = \boldsymbol{\Gamma^{1}_{00,1}} + \Gamma^0_{01}\,\Gamma^1_{00} + \Gamma^1_{00}\,\Gamma^0_{10}$$

After substitution and simplification:

$$R_{00} = \frac{-R_s(3R_s - 2r)}{2r^4} + \frac{R_s}{2r^2\sigma^2}\,\frac{\sigma^2 R_s}{2r^2} + \frac{\sigma^2 R_s}{2r^2}\frac{R_s}{2r^2\sigma^2}$$

$$R_{00} = \frac{-R_s(3R_s - 2r)}{2r^4} + \frac{{R_s}^2}{2r^4} =$$

$$R_{00} = \frac{-R_s(2R_s - 2r)}{2r^4} = \frac{-R_s(R_s - r)}{r^4}$$

Conclusion:

$$\boldsymbol{R_{00} \neq 0}$$

Component R_{11}

$$R_{11} = \boldsymbol{\Gamma^{1}_{11,1}} + \Gamma^0_{10}\Gamma^0_{01} + \Gamma^1_{11}\,\Gamma^1_{11} + \Gamma^2_{12}\,\Gamma^2_{21} + \Gamma^3_{13}\,\Gamma^3_{31}$$

Elaboration leads to:

$$R_{11} = \frac{-R_s(2r - Rs)}{2r^4\sigma^4} + \frac{R_s}{2r^2\sigma^2}\frac{R_s}{2r^2\sigma^2} + \frac{-R_s}{2r^2\sigma^2}\,\frac{-R_s}{2r^2\sigma^2} + \frac{1}{r}\,\frac{1}{r} + \frac{1}{r}\,\frac{1}{r}$$

$$R_{11} = \frac{-R_s(2r - Rs)}{2r^4\sigma^4} + \frac{{R_s}^2}{2r^4\sigma^4} + \frac{2}{r^2}$$

$$R_{11} = \frac{-R_s(2r - Rs)}{2r^4\sigma^4} + \frac{{R_s}^2}{2r^4\sigma^4} + \frac{4\left(r^2 + {R_s}^2 - 2rR_s\right)}{2r^4\sigma^4}$$

$$R_{11} = \frac{-2rR_s + {R_s}^2}{2r^4\sigma^4} + \frac{{R_s}^2}{2r^4\sigma^4} + \frac{4\left(r^2 + {R_s}^2 - 2rR_s\right)}{2r^4\sigma^4}$$

$$R_{11} = \frac{-2rR_s + 2{R_s}^2}{2r^4\sigma^4} + \frac{4\left(r^2 + {R_s}^2 - 2rR_s\right)}{2r^4\sigma^4}$$

$$R_{11} = \frac{-2rR_s + 2{R_s}^2 + 4r^2 + 4{R_s}^2 - 8rR_s}{2r^4\sigma^4}$$

$$R_{11} = \frac{-10rR_s + 6{R_s}^2 + 4r^2}{2r^4\sigma^4} = \frac{3{R_s}^2 + 2r^2 - 5rR_s}{r^4\sigma^4}$$

Through further simplification:

$$R_{11} = \frac{-10rR_s + 6{R_s}^2 + 4r^2}{2r^4\sigma^4} = \frac{3{R_s}^2 + 2r^2 - 5rR_s}{r^2({R_s}^2 + r^2 - 2rR_s)}$$

Conclusion:

$$\boldsymbol{R_{11} \neq 0}$$

Component $\boldsymbol{R_{22}}$

$$R_{22} = \boldsymbol{\Gamma^{1}_{22,1}} + \boldsymbol{\Gamma^{2}_{22,2}} + \Gamma^{1}_{22}\,\Gamma^{2}_{12} + \Gamma^{2}_{21}\,\Gamma^{1}_{22} + \Gamma^{2}_{22}\,\Gamma^{2}_{22} + \Gamma^{3}_{23}\,\Gamma^{3}_{23}$$

Evaluation of these terms gives:

$$R_{22} = 1 + 0 + (-r\sigma^2)\,\frac{1}{r} + \frac{1}{r}\,(-r\sigma^2) + 0\ + \frac{\cos\theta}{\sin\theta}\,\frac{\cos\theta}{\sin\theta}$$

$$R_{22} = 1 - 2\sigma^2\ + \frac{\cos^2\theta}{\sin^2\theta} = \frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} - 2\sigma^2$$

$$R_{22} = \frac{1}{\sin^2\theta} - 2\sigma^2$$

Conclusion:

$$\boldsymbol{R_{22} \neq 0}$$

Component $\boldsymbol{R_{33}}$

$$R_{33} = +\boldsymbol{\Gamma^{1}_{33,1}} + \boldsymbol{\Gamma^{2}_{33,2}} + \Gamma^{1}_{33}\,\Gamma^{3}_{13} + \Gamma^{2}_{33}\,\Gamma^{3}_{23} + \Gamma^{3}_{31}\,\Gamma^{1}_{33} + \Gamma^{3}_{32}\,\Gamma^{2}_{33}$$

$$R_{33} = 1 + \cos^2\theta - \sin^2\theta\ \ - r\sigma^2\sin^2\theta\,\frac{1}{r} - \cos\theta\sin\theta.\,\frac{\cos\theta}{\sin\theta} + \frac{1}{r}\,(-r\sigma^2\sin^2\theta) + \frac{\cos\theta}{\sin\theta}\,(-\cos\theta\sin\theta)$$

$$R_{33} = 1 + \cos^2\theta - \sin^2\theta\ \ - 2\sigma^2\sin^2\theta\ - 2\cos\theta\sin\theta.\,\frac{\cos\theta}{\sin\theta}$$

$$R_{33} = 1 + \cos^2\theta - \sin^2\theta\ \ - 2\sigma^2\sin^2\theta\ - 2\cos^2\theta$$

$$R_{33} = 1 - \cos^2\theta - \sin^2\theta\ \ - 2\sigma^2\sin^2\theta$$

$$R_{33} = -2\sigma^2\sin^2\theta$$

Conclusion:

$$\boldsymbol{R_{33} \neq 0}$$

General Conclusion

We see that all components are $R_{\mu\nu}\ \neq 0$ when using the *restricted* Einstein field equations. Thus, the Schwarzschild formula with spherical/polar coordinates does not satisfy this *limited* formula. This is not surprising, since the determinant of g for the spherical coordinates is not -1, which is a requirement for using the limited formula. For the Schwarzschild metric in spherical coordinates, the following applies:

$$g = -\sigma^2 \cdot \frac{1}{\sigma^2} \cdot r^2 \cdot r^2 \sin^2\theta = -r^4 \sin^2\theta \neq -1$$

However, with regard to the full formula for the Einstein field equations, Schwarzschild's spherical/polar coordinate equation is consistent, as demonstrated above.

Note:

The restricted formula was the result of the additional condition that Einstein added, namely that the product of the elements of the trace of the metric tensor must be *g*=-1 ($g = g_{00} . g_{11} . g_{22} . g_{33} = -1$). This additional condition was introduced by Einstein to simplify the calculations and his general formula. However, the restricted formula is a restriction that ignores a number of possible solutions. Therefore, in my opinion, applying the general formula is the best approach. This is also supported by the fact that the practical Schwarzschild equation:

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta \, d\emptyset^2$$

has a *g* that is not equal to -1 and therefore does not satisfy the restricted Einstein formula, but does satisfy the general formula. This formula can be used to solve all kinds of practical problems in general relativity, such as the bending of light, the orbit of Mercury, Shapiro time delay, and so on. Various measurements have also shown that the calculations correspond to the measurement results. In short, Schwarzschild's solution shows that the general formula of general relativity theory is preferred over the restricted formula.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

is preferable to the limited formula.

The general Einstein field equations form the correct basis of relativistic gravitational theory. The limited formula is only a specialization under restrictive conditions.

Part VII – Questions and Discussion

7 Answers to Questions

7.1 Derivation of the Schwarzschild Formula to proper time τ

Question:
What I find difficult to accept in general relativity is the derivation to "*ds*." The line element is nothing more than the speed of light multiplied by the locally measured time difference "dt_0" ($ds = c.\ dt_0$). I can understand *dt/ds* (difference in clock times), but what does *dx/ds* mean?

Answer:

The confusion arises from the interpretation of *ds*. In general relativity, the following applies:

$$ds = cd\tau$$

Where τ is the **proper time**: the time measured by a clock that is at rest relative to the object being measured—in other words, the time on a clock that "moves" with the object itself.

The quantity *dt*, on the other hand, is the coordinate time in a universal (or external) reference frame, for example the center of a gravitational field (such as the center of the Earth). This time *t* is therefore not a directly measured time, but a mathematical parameter that can be derived from$d\tau$ via the metric.

The relationship between the two is:

$$d\tau = \frac{\sigma}{\gamma} dt$$

In this,

- $\sigma = \sqrt{1 - \frac{R_s}{r}}$ the gravitational factor (derived from the Schwarzschild solution),
- $\gamma = \frac{1}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$ the Lorentz factor,
- $R_s = \frac{2\text{GM}}{c^2}$ the Schwarzschild radius.

Derivation from the Schwarzschild metric

We consider the time interval based on the general form of the metric tensor product in a stationary, spherically symmetric field:

$$c^2 d\tau^2 = Ac^2 dt^2 - Bdx^2 - Ddy^2 - Edz^2$$

where *A, B, D, E* are the components of the metric tensor, depending on the position in space (e.g., on *r*).

Divide both sides by $c^2 d\tau^2$:

$$1 = A\left(\frac{dt}{d\tau}\right)^2 - \frac{B}{c^2}\left(\frac{dx}{d\tau}\right)^2 - \frac{D}{c^2}\left(\frac{dy}{d\tau}\right)^2 - \frac{E}{c^2}\left(\frac{dz}{d\tau}\right)^2$$

We rewrite the spatial derivatives using the chain rule:

$$\frac{dx}{d\tau} = \frac{dx}{dt} \cdot \frac{dt}{d\tau}, \quad \text{etc.}$$

This gives the equation:

$$1 = A\left(\frac{dt}{d\tau}\right)^2 - \frac{B}{c^2}\left(\frac{dx}{dt}\right)^2\left(\frac{dt}{d\tau}\right)^2 - \frac{D}{c^2}\left(\frac{dy}{dt}\right)^2\left(\frac{dt}{d\tau}\right)^2 - \frac{E}{c^2}\left(\frac{dz}{dt}\right)^2\left(\frac{dt}{d\tau}\right)^2$$

Here, *x, y*, and *z* are divided by *t* in their own frame and turn out to be velocities in that frame.

$$1 = A\left(\frac{dt}{d\tau}\right)^2 \left\{1 - \frac{B}{Ac^2}\left(\frac{dx}{dt}\right)^2 - \frac{D}{Ac^2}\left(\frac{dy}{dt}\right)^2 - \frac{E}{Ac^2}\left(\frac{dz}{dt}\right)^2\right\}$$

$$v^2 = \frac{B}{A}\left(\frac{dx}{dt}\right)^2 - \frac{D}{A}\left(\frac{dy}{dt}\right)^2 - \frac{E}{A}\left(\frac{dz}{dt}\right)^2$$

Now substitute$v$ into the previous equation:

$$1 = A\left(\frac{dt}{d\tau}\right)^2\left(1 - \frac{v^2}{c^2}\right) = \frac{A}{\gamma^2}\left(\frac{dt}{d\tau}\right)^2$$

This results in:

$$\left(\frac{dt}{d\tau}\right)^2 = \frac{\gamma^2}{A} \quad \Longrightarrow \quad d\tau^2 = \frac{A}{\gamma^2}dt^2 = \frac{\sigma^2}{\gamma^2}dt^2$$

Or:

$$d\tau = \frac{\sigma}{\gamma}dt$$

This is the relationship between the time of the measuring clock and the time at the origin of the universal frame.

Conclusion

This derivation shows the relationship between the proper timeτ (as measured by a moving clock in its own rest frame) and the coordinate time *t* (as defined in the global gravitational field). The role of *dx/ds* also becomes clear: it describes the speed of spatial change per unit of proper time—i.e., the projection of the four-velocity onto the spatial coordinates.

This relationship is fundamental in general relativity and forms the basis for analyzing time dilation in gravitational fields, such as in the Hafele–Keating experiment and other applications of the Schwarzschild metric.

7.2 Explanation of Einstein's Transformation Formula

In general relativity, it is fundamental that physical laws remain invariant under coordinate transformations. The relationship between old and new coordinate systems is expressed mathematically using a transformation formula based on partial derivatives.

1. Coordinate systems

The formula represents the covariant transformation between two coordinate systems. The old system is denoted by x_β , i.e., with coordinate axes x_0, x_1, x_2, x_3 . The new system is denoted by $x_\alpha^{'}$, with $x_0^{'}, x_1^{'}, x_2^{'}, x_3^{'}$.

2. Transformation formula

We express the differential of the new coordinates $dx_\alpha^{'}$ in terms of the differential of the old coordinates dx_β as follows:

$$dx_\alpha^{'} = \frac{\partial x_\beta}{\partial x_\alpha^{'}} dx_\beta$$

This formula is written according to Einstein notation, which means that there is a summation over β .
This actually means:

$$dx_\alpha^{'} = \frac{\partial x_0}{\partial x_\alpha^{'}} dx_0 + \frac{\partial x_1}{\partial x_\alpha^{'}} dx_1 + \frac{\partial x_2}{\partial x_\alpha^{'}} dx_2 + \frac{\partial x_3}{\partial x_\alpha^{'}} dx_3$$

For each value of α (0 to 3), this gives a separate equation that expresses each of the new coordinate differentials $dx_0^{'}, dx_1^{'}, dx_2^{'}, dx_3^{'}$ in terms of the old coordinates.

3. Tensor form

In total, we then get:

$$dx_0^{'} = \frac{\partial x_0}{\partial x_0^{'}} dx_0 + \frac{\partial x_1}{\partial x_0^{'}} dx_1 + \frac{\partial x_2}{\partial x_0^{'}} dx_2 + \frac{\partial x_3}{\partial x_0^{'}} dx_3$$

$$dx_1^{'} = \frac{\partial x_0}{\partial x_1^{'}} dx_0 + \frac{\partial x_1}{\partial x_1^{'}} dx_1 + \frac{\partial x_2}{\partial x_1^{'}} dx_2 + \frac{\partial x_3}{\partial x_1^{'}} dx_3$$

$$dx_2^{'} = \frac{\partial x_0}{\partial x_2^{'}} dx_0 + \frac{\partial x_1}{\partial x_2^{'}} dx_1 + \frac{\partial x_2}{\partial x_2^{'}} dx_2 + \frac{\partial x_3}{\partial x_2^{'}} dx_3$$

$$dx_3^{'} = \frac{\partial x_0}{\partial x_3^{'}} dx_0 + \frac{\partial x_1}{\partial x_3^{'}} dx_1 + \frac{\partial x_2}{\partial x_3^{'}} dx_2 + \frac{\partial x_3}{\partial x_3^{'}} dx_3$$

This can also be represented as a tensor (matrix form):

$$\begin{pmatrix} dx_0' \\ dx_1' \\ dx_2' \\ dx_3' \end{pmatrix} = \begin{bmatrix} \frac{\partial x_0}{\partial x_0'} & \frac{\partial x_1}{\partial x_0'} & \frac{\partial x_2}{\partial x_0'} & \frac{\partial x_3}{\partial x_0'} \\ \frac{\partial x_0}{\partial x_1'} & \frac{\partial x_1}{\partial x_1'} & \frac{\partial x_2}{\partial x_1'} & \frac{\partial x_3}{\partial x_1'} \\ \frac{\partial x_0}{\partial x_2'} & \frac{\partial x_1}{\partial x_2'} & \frac{\partial x_2}{\partial x_2'} & \frac{\partial x_3}{\partial x_2'} \\ \frac{\partial x_0}{\partial x_3'} & \frac{\partial x_1}{\partial x_3'} & \frac{\partial x_2}{\partial x_3'} & \frac{\partial x_3}{\partial x_3'} \end{bmatrix} \cdot \begin{pmatrix} dx_0 \\ dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$$

This matrix represents the **Jacobian of the coordinate transformation** and describes how vector components transform between two systems.

4. Interpretation

This formula shows how a vector (or differential) in one system can be expressed in the other system. Important here is:

- The **transformation factors** $\frac{\partial x_\alpha}{\partial x_\beta'}$ indicate how the old coordinates change with respect to the new ones.
- In tensor analysis, we speak of a **covariant transformation** if the indices are on the right (such as ∂x_β) and **a contravariant transformation** if the indices are above (such as dx^β).

5. Example: Transformation within Schwarzschild metric

A practical application is the transition from spherical coordinates (t, r, θ, φ) to Cartesian coordinates (*t, x, y, z*). Here, the spatial coordinates are transformed via:

$$x = r \sin\theta \cos\varphi$$

$$x = r \sin\theta \sin\varphi$$

$$x = r \cos\theta$$

The corresponding differential transformations for *dx, dy, dz* can then be derived using the chain rule, as formalized above.

7.3 Answer to questions concerning Schwarzschild

Question 1:

Where does the general relativity formula with the Ricci tensor come from, which only came into general use after 1916?

Answer:

The complete field equations of general relativity, including the Ricci tensor, were part of Einstein's theory from the beginning. The simplified version with the condition g=−1 was later used to make the equations mathematically simpler, but this restriction reduces the number of possible solutions.

In much of the literature, the tensor $G_{\mathrm{\mu\nu}}$ is called the **Einstein tensor**. Einstein himself presented this tensor as:

$$G_{\mathrm{\mu\nu}} = R_{\mathrm{\mu\nu}} - \frac{1}{2} g_{\mathrm{\mu\nu}} R$$

In this,

- $R_{\mathrm{\mu\nu}}$: the **Ricci tensor**,
- R: the **Ricci scalar**, or the trace of the Ricci tensor,
- $g_{\mathrm{\mu\nu}}$: the **metric tensor** of space-time.

The Ricci scalar is given by:

$$R = g^{\mu\nu} R_{\mu\nu} = g^{00} R_{00} + g^{11} R_{11} + g^{22} R_{22} + g^{33} R_{33}$$

Contraction of the Einstein field equations with $g^{\mu\nu}$ yields:

$$g^{\mu\nu} G_{\mathrm{\mu\nu}} = g^{\mu\nu} R_{\mathrm{\mu\nu}} - \frac{1}{2} g^{\mu\nu} g_{\mathrm{\mu\nu}} R = R - \frac{1}{2} 4R = -R$$

The **complete Einstein field equations** are:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Where:

- $R_{\mu\nu}$: the Ricci tensor,
- $g_{\mu\nu}$: the metric tensor,
- *G:* the gravitational constant,
- $T_{\mu\nu}$: the energy-momentum tensor,
- c: the speed of light.

There is no matter or energy outside a massive sphere. In that case, $T_{\mathrm{\mu\nu}} = 0$, and the field equations simplify to:

$$G_{\mathrm{\mu\nu}} = R_{\mathrm{\mu\nu}} - \frac{1}{2} g_{\mathrm{\mu\nu}} R = 0$$

We know that:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\mu\nu} R_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} 4R_{\mu\nu} = -R_{\mu\nu}$$

From which it follows:

$$\boldsymbol{G_{\mu\nu} = 0} \quad \textbf{only if} \quad \boldsymbol{R_{\mu\nu} = 0} \quad \textbf{and also} \quad \boldsymbol{R = 0}$$

Background: from Riemann to Ricci

Einstein built on the work of Riemann, who had already developed a mathematical description of curved surfaces. The **Riemann tensor**:

$$R_{\mu\beta\rho\nu}$$

This is a rank-four tensor and difficult to visualize. Because the mass-energy-momentum tensor $T_{\mu\nu}$ has only two indices, the Riemann tensor must be converted from four to two indices.

Using the metric tensor, the covariant Riemann tensor can be converted into a partially contravariant form:

$$R^{\beta}{}_{\mu\rho\nu} = g^{\beta\beta} R_{\mu\beta\rho\nu}$$

This is necessary to perform the desired contraction. By setting $\beta = \rho$, the contraction can be performed, resulting in the Ricci tensor $R_{\mu\nu}$.

$$R^{\beta}{}_{\mu\beta\nu} = R_{\mu\nu}$$

Here, the Ricci tensor is the **trace** of the Riemann tensor.

The role of Christoffel symbols

The Ricci tensor can also be expressed in terms of **Christoffel symbols**:

$$\mathrm{R}_{\mu\nu} = \mathrm{R}^{\rho}_{\mu\rho\nu} = \Gamma^{\rho}_{\mu\nu,\rho} - \Gamma^{\rho}_{\rho\mu,\nu} + \Gamma^{\rho}_{\rho\lambda} \Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\rho\mu}$$

where the Christoffel symbol itself is:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\}$$

The derivative of the Christoffel symbol then becomes:

$$\Gamma^{\rho}_{\mu\nu,\gamma} = \frac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\gamma}} = -g^{\rho\alpha} \cdot \frac{\partial g_{\rho\alpha}}{\partial x^{\gamma}} \cdot \Gamma^{\rho}_{\mu\nu} + \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial^2 g_{\nu\alpha}}{\partial x^{\mu} \partial x^{\gamma}} + \frac{\partial^2 g_{\mu\alpha}}{\partial x^{\nu} \partial x^{\gamma}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha} \partial x^{\gamma}} \right\}$$

When calculating the Ricci elements $R_{00}, R_{11}, R_{22}, R_{33}$ with the complete Einstein field equations, it appears that these are all zero, which is correct. But when we do the calculation with the limited formula of the field equations (g=-1), the result is not correct. So the Schwarzschild equation does satisfy the general field equations but not the limited ones. This is correct because in the Schwarzschild equation $g \neq -1$.

About the Schwarzschild solution and the restriction g=−1

Schwarzschild uses the well-known polar equation. The determinant of the metric tensor (here the product of the coefficients) is not -1. This polar equation satisfies the Einstein field equations, but not the restricted version of these equations, because the latter requires g=− 1. Schwarzschild derived a transformation based on modified polar coordinates, choosing the transformation such that g=− 1 is achieved. In that case, the equation also satisfies the restricted Einstein field equations.

Conclusion

Although Schwarzschild attempted to satisfy Einstein's desire to have the metric trace *g*=− 1, in my opinion, the only relevant question is whether the Einstein field equations, where $T_{\mu\nu} = 0$, and thus $R_{00} = R_{11} = R_{22} = R_{33} = 0$, are satisfied, regardless of whether $g = -1$ of $g \neq -1$. Thus, the requirement of $g = -1$ is an unnecessary restriction.

Question 2:

The difference in formulas has significant consequences. In your document, I count nine Christoffel symbols, while Karl Schwarzschild found ten. In your case, the 222 (Γ_{22}^{2}) seems to be absent. This is because your definition of the metric tensor g differs from that of Schwarzschild; g_{22} and g_{33} are -1 for Schwarzschild, while you add the coordinate r (for example, $g_{22} = -r^2$). Droste (1917), Eddington (1921), MWT (1975), and OAS (2007) also adhered to g=-1 for the Schwarzschild solution, so that: $g_{22} = g_{33} = -1$. This raises the question for me: do you think g=-1 is required for the Schwarzschild solution?

Answer:

Schwarzschild's original derivation started from the Cartesian coordinate system (*x, y, z*). The metric in that form has the following components:

$$g_{00} = \sigma^2 \quad g_{11} = -\frac{1}{r^4\sigma^2} \quad g_{22} = -\frac{r^2}{\sin^2\theta} \quad g_{33} = -r^2\sin^2\theta$$

In this case, ten (or fourteen) relevant Christoffel symbols are created. You can also see in my overview of formulas that I have derived formulas for both the spherical and the Cartesian *x, y, z* form. In the *x, y, z* form, 222 (Γ_{22}^{2}) .

However, this is different for the spherical form; there, the elements of the metric tensor are:

$$g_{00} = \sigma^2 \quad g_{11} = \frac{-1}{\sigma^2} \quad g_{22} = -r^2 \quad g_{33} = -r^2\sin^2\theta$$

This is exactly the same form as in Schwarzschild. In these spherical coordinates, g_{22} and g_{33} explicitly depend on$r$ and θ , and therefore cannot be constant, as in$g_{22} = g_{33} = -1$. If these values were constant, the partial derivatives $\frac{\partial g22}{\partial r}, \frac{\partial g22}{\partial \theta}, \frac{\partial g33}{\partial r},$ and $\frac{\partial g33}{\partial \theta}$ would all be zero. As a result, many Christoffel symbols, including crucial ones such as Γ_{22}^{1} and Γ_{22}^{2} , would also disappear.

This also applies to Schwarzschild! The elements g_{22} en g_{33} cannot be -1 because in that case $\frac{\partial g_{22}}{\partial r}, \frac{\partial g_{22}}{\partial \theta}, \frac{\partial g_{33}}{\partial r}, \frac{\partial g_{33}}{\partial \theta}$ would be zero and the number of Christoffel symbols would be limited to 001 (and 010), 100, and 111.

As for Γ_{22}^{2} :
For spherical coordinates, this component is indeed zero, because g_{22} does not depend onθ and the derivative is therefore zero:

$$\Gamma_{22}^{2} = \frac{1}{2} g^{22} \left(\frac{\partial g_{22}}{\partial \theta}\right) = 0$$

It is important to note that when evaluating components for a specific value such as $\theta = \frac{\pi}{2}$ (or90^0), substitution should only be performed at the **end** of the calculations.
For example:

$$\Gamma_{33}^{2} = \frac{1}{2} g^{22} \left\{ -\frac{\partial g_{33}}{\partial \theta} \right\} = -\boldsymbol{\cos\theta \sin\theta = 0} \ \text{ when } \theta = 90^0$$

This value becomes zero when $\theta = \frac{\pi}{2}$, but for the Ricci element, the **derivative** of this Christoffel symbol with respect to θ is also needed, which is:

$$\frac{\partial \Gamma_{33}^{2}}{\partial \theta} = \boldsymbol{-\cos^2\theta + \sin^2\theta = 1} \ \text{ when } \theta = 90^0$$

And that is **not zero**, which is crucial for calculating, for example, the Ricci tensor component R_{22} .

Regarding the condition det(g)=−1:

Why Einstein introduced this restriction is not entirely clear, but it simplifies the algebra in many cases and ensures symmetry. However, in my opinion, this leads to an unnecessary limitation. It also depends on which type of coordinate system is chosen. For example, the element of the metric tensor of *t, x, y, z* does indeed yield a det(g) of -1:

$$\sigma^2 . \left(-\frac{1}{r^4 \sigma^2} \right) . \left(-\frac{r^2}{\sin^2\theta} \right) . (-r^2 \sin^2\theta) = -1$$

But with spherical coordinates, it is:

$$\sigma^2 . \frac{-1}{\sigma^2} . (-r^2) . (-r^2 \sin^2\theta) = - \ r^4 \sin^2\theta$$

And so here, $\det(g) \neq -1$. Nevertheless, this metric perfectly satisfies the Einstein field equations in vacuum $\left(T_{\mu\nu} = 0\right)$, which means that $R_{\mu\nu} = 0$ and therefore also R=0.

Conclusion:
The requirement det(g)=−1 is a **coordinate-dependent convention** that may offer mathematical convenience, but is **not** physically **necessary** for the correctness of the Schwarzschild solution. What really matters is that the field equations are satisfied. Schwarzschild's choice to use a transformation that achieves det(g*)*=−1 was primarily intended to satisfy Einstein's wishes, but is unnecessary from a physical point of view.

Question 3:

The field equations in your document on pages 2 and 3, based on the Ricci tensor, differ greatly from those we (and Karl Schwarzschild) used in Appendix E, based on the G tensor. You also mentioned the G tensor in your document on page 9. My question is: shouldn't the results be the same?

Answer:

By the G-tensor, as you mention in your question, you mean Einstein's restricted field equations. As I have shown theoretically in my story above, Schwarzschild satisfies the general field equations but not the restricted field equations G. This is because Einstein introduced the additional requirement of det(g*)*=-1 in order to obtain simpler formulas, but this led to unnecessary restrictions on possible solutions such as the Schwarzschild equation based on spherical coordinates. While this equation is a great solution for calculating phenomena in a vacuum in a reasonably simple way.

Question 4:

I still have some difficulty understanding the Schwarzschild equation and Einstein's field equations. Could you elaborate on this?

Answer:

It seems that we are re-entering a discussion we have had before. Let me make it clear from the outset: it is not my intention to defend the Schwarzschild or Einstein solution against your approach, or to criticize your suggestion to modify the Schwarzschild metric. My goal is to achieve a complete understanding. As long as I do not fully understand the Schwarzschild solution, I will continue to seek insight. Only when I recognize and understand a fundamental error will I consider revising the solution.

Let us therefore first examine the Schwarzschild solution in detail before delving into the Einstein field equations. I do not claim to already know the complete answer, but I would like to explain how I understand the structure so far.

Einstein's starting point

Einstein sought a description of gravity in which gravity is **no** longer **a force**, as in Newton's theory, but rather a consequence of the **curvature of space-time**. He wanted to find a coordinate system in which no forces are felt, so that a freely falling particle moves without acceleration—in a sense, "voluntarily," without any external cause.

In classical mechanics, an object moves at a constant speed along a straight line if no force acts on it. Einstein translated this into the theory of relativity: an object without external force moves along a **geodesic line** in curved space-time. These geodesics are, in a sense, the "straight lines" of curved space-time.

Einstein therefore sought a mathematical formulation that would be valid for **any coordinate system**, curved or not, and would correctly describe the gravitational field. This led to the field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

in the vacuum case (outside a mass), where:

- $R_{\mu\nu}$ the **Ricci tensor** is a measure of local curvature,
- R the **Ricci scalar,** the trace of the Ricci tensor,
- $g_{\mu\nu}$ the **metric,** which describes the space-time structure.

This equation is **covariant**: it is valid in any coordinate system.

Coordinates, metric, and geometry

Although the field equations are coordinate-independent, the components of the tensors involved are dependent on the choice of coordinate system. The Ricci tensor and the scalar R are expressed in terms of the **Christoffel symbols**, which are themselves derived from the **metric** $g_{\mu\nu}$.

The metric describes how the distance ds^2 between two infinitesimally close points is calculated. In the simplest case (e.g., flat space in Cartesian coordinates), this is:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

But in curved space-time, ds^2 depends on the location and on the metric components. The metric contains information about the geometric structure of space, including possible **cross terms** (such as *dx dy*) if the coordinates are not orthogonal.

For comparison:

- In a rectangular flat space, $c^2 = a^2 + b^2$ applies (Pythagoras).
- In an oblique space, the following applies: $c^2 = a^2 + b^2 - 2ab\cos\varphi$ (cosine rule).

By analogy, Einstein considers space-time to be composed of an infinite number of **infinitesimally small planes**, on which the geometry can be considered locally as a plane (via the equivalence principle). In these small areas, we still use a coordinate system, but the metric components change from location to location—and encode the curvature.

Schwarzschild's approach

Karl Schwarzschild found an exact solution to Einstein's equations in a vacuum around a spherical mass. He chose a **coordinate system** that contains as much symmetry as possible:

- spherically symmetric,
- static (time-independent),
- and without cross terms (i.e., a diagonal metric).

This yields the **Schwarzschild metric**:

$$ds^2 = \left(1 - \frac{2GM}{c^2r}\right)c^2dt^2 - \left(1 - \frac{2GM}{c^2r}\right)^{-1} dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\emptyset^2$$

This formula describes the quadratic line segment ds^2 as a function of four coordinates: $(t, r, \theta, \emptyset)$. The coefficients (metric components) depend on *r* (and implicitly, via $\sin^2\theta$, also on θ , but not on t or $\emptyset$.

This reflects the physical assumptions: the solution is static (time-independent) and spherically symmetric.

Each location in space has its own metric components, and therefore its own geometric structure. By integrating *ds* along a path, we obtain the total distance or duration of the trajectory in this curved space-time.

Summary

- The Einstein field equations describe how mass and energy determine the **curvature** of space-time.
- The Ricci tensor and Einstein tensor are geometric objects that are independent of the coordinate system, but their **components** change when a different coordinate system is chosen.
- Schwarzschild chose a specific coordinate system and found an exact solution for the gravitational field outside a spherically symmetric mass.
- His solution is consistent with Einstein's equations and is still one of the most important solutions in general relativity.

7.4 Detailed derivation of the Einstein Equation (57) from equation (53)

Question:

I am reading Einstein's original GR paper. I have attached it as a PDF to this email. (Einstein, Relativity: The Special and General Theory, 1916 (this revised edition: 1924)) (Einstein, The Collected Papers of Albert Einstein, 1997)
In section 18, at the bottom of page 186 of the article (bottom left of page 22 of the PDF), there is an equation that I am trying to derive using the method Einstein proposes in the article (multiplying equation 53 by the derivative of the metric tensor and using the methods in section 15). Could you derive this equation in the specific way Einstein indicates, based solely on the preceding material in Einstein's article? Can you show me the detailed steps you took to arrive at that equation according to the method Einstein indicates?

Answer:

Note: the equation numbers refer to Einstein's original work on General Relativity.

Derivation of equation (57) from equation (53)

We start from equation (53) in Einstein's article:

$$\frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} = -\kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$$

Here, the following applies:

- $T_{\mu\nu}$ is the energy-momentum tensor,
- $T = g^{\mu\nu}T_{\mu\nu}$ is its trace,
- and we assume that $\sqrt{-g} = 1$, as Einstein does in section 18.

Step 1: Multiply by $\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}$

We multiply both sides of equation (53) by $\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}$, or:

$$\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}\left(\frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}}+\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\right)=\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}\left(-\kappa\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right)\right)=$$

This leads to:

$$=-\kappa\left(\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}T\right)$$

Step 2: Use that $\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}g_{\mu\nu}=0$

From equation (29) in Einstein's article, it follows that:

$$\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x^{\sigma}}=-\frac{1}{2}g_{\mu\nu}\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}$$

Because $\sqrt{-g}=-1$, its derivative is zero. So:

$$-\frac{1}{2}g_{\mu\nu}\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}=0$$

Filling in, we get:

$$=-\kappa\left(\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}T\right)=-\kappa\left(\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}T_{\mu\nu}-0\right)=-\kappa\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}T_{\mu\nu}$$

Then the equation becomes:

$$\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}\frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}}+\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}+\kappa\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}T_{\mu\nu}=0$$

The next step gives:

$$\frac{1}{2\kappa}\left(\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}\frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}}+\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\right)+\frac{1}{2}\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}T_{\mu\nu}=0 \qquad (1)$$

Substitute this into:

$$\frac{1}{2\kappa}\left(\frac{\partial}{\partial x^{\alpha}}(-2\kappa t^{\alpha}_{\sigma})\right)+\frac{1}{2}\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}T_{\mu\nu}=0$$

See the calculation of the yellow part below the dotted line below.

This leads to:

$$-\frac{\partial t^{\alpha}_{\sigma}}{\partial x^{\alpha}}+\frac{1}{2}\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}T_{\mu\nu}=0 \qquad (2)$$

Now use Einstein's equation (56):

$$\boldsymbol{\frac{\partial\left(t^{\sigma}_{\mu}+T^{\sigma}_{\mu}\right)}{\partial x^{\sigma}}=0}$$

So:

$$\frac{\partial t_\mu^\sigma}{\partial x^\sigma} = -\frac{\partial T_\mu^\sigma}{\partial x^\sigma}$$

Replace σ door α, en μ door σ:

$$\frac{\partial t_\sigma^\alpha}{\partial x^\alpha} = -\frac{\partial T_\sigma^\alpha}{\partial x^\alpha}$$

Equation (2) becomes:

$$\boldsymbol{\frac{\partial T_\sigma^\alpha}{\partial x^\alpha} + \frac{1}{2}\frac{\partial g^{\mu\nu}}{\partial x^\sigma} T_{\mu\nu} = 0} \qquad \mathbf{(57)}$$

This is precisely Einstein's equation (57).

===

Derivation of the yellow step:

To prove that:

$$\frac{\partial}{\partial x^\alpha}(-2\kappa t_\sigma^\alpha) = \frac{\partial g^{\mu\nu}}{\partial x^\sigma}\frac{\partial \Gamma_{\mu\nu}^\alpha}{\partial x^\alpha} + \frac{\partial g^{\mu\nu}}{\partial x^\sigma}\Gamma_{\mu\beta}^\alpha \Gamma_{\nu\alpha}^\beta$$

We use Einstein's equation (48):

$$\frac{\partial H}{\partial g^{\mu\nu}} = -\Gamma_{\mu\beta}^\alpha \Gamma_{\nu\alpha}^\beta$$

$$\frac{\partial H}{\partial g_\sigma^{\mu\nu}} = \Gamma_{\mu\nu}^\sigma$$

Einstein equation (47b):

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial H}{\partial g_\alpha^{\mu\nu}}\right) - \frac{\partial H}{\partial g^{\mu\nu}} = 0 \Rightarrow \frac{\partial}{\partial x^\alpha}\left(\frac{\partial H}{\partial g_\alpha^{\mu\nu}}\right) = \frac{\partial H}{\partial g^{\mu\nu}}$$

$$\frac{\partial g^{\mu\nu}}{\partial x^\sigma}\frac{\partial \Gamma_{\mu\nu}^\alpha}{\partial x^\alpha} + \frac{\partial g^{\mu\nu}}{\partial x^\sigma}\Gamma_{\mu\beta}^\alpha \Gamma_{\nu\alpha}^\beta$$

Can be rewritten as:

$$\frac{\partial g^{\mu\nu}}{\partial x^\sigma}\frac{\partial}{\partial x^\alpha}\left(\frac{\partial H}{\partial g_\alpha^{\mu\nu}}\right) - \frac{\partial g^{\mu\nu}}{\partial x^\sigma}\frac{\partial H}{\partial g^{\mu\nu}}$$

Now we can differentiate:

$$\frac{\partial}{\partial x^\alpha}\left(g_\sigma^{\mu\nu}\frac{\partial H}{\partial g_\alpha^{\mu\nu}}\right) - \frac{\partial\left(g_\sigma^{\mu\nu}\right)}{\partial x^\alpha}\frac{\partial H}{\partial g_\alpha^{\mu\nu}} - \frac{\partial g^{\mu\nu}}{\partial x^\sigma}\frac{\partial H}{\partial g^{\mu\nu}}$$

Here, the following applies:

$$\frac{\partial\left(g_\sigma^{\mu\nu}\right)}{\partial x^\alpha} = \frac{\partial^2 g^{\mu\nu}}{\partial x^\alpha \partial x^\sigma} = \frac{\partial\left(g_\alpha^{\mu\nu}\right)}{\partial x^\sigma}$$

Fill this in:

$$\frac{\partial}{\partial x^\alpha}\left(g_\sigma^{\mu\nu}\frac{\partial H}{\partial g_\alpha^{\mu\nu}}\right)-\frac{\partial\left(g_\alpha^{\mu\nu}\right)}{\partial x^\sigma}\frac{\partial H}{\partial g_\alpha^{\mu\nu}}-\frac{\partial H}{\partial g^{\mu\nu}}\frac{\partial g^{\mu\nu}}{\partial x^\sigma}$$

As stated in Einstein's document under equation (47a), *H* is considered a function of $g^{\mu\nu}$ and $g_\sigma^{\mu\nu}$ $\left(=\frac{\partial g^{\mu\nu}}{\partial x^\sigma}\right)$, so:

$$\frac{\partial H}{\partial x^\sigma}=\frac{\partial H}{\partial g_{,\alpha}^{\mu\nu}}\frac{\partial g_{,\alpha}^{\mu\nu}}{\partial x^\sigma}+\frac{\partial H}{\partial g^{\mu\nu}}\frac{\partial g^{\mu\nu}}{\partial x^\sigma}$$

Fill this in:

$$\frac{\partial}{\partial x^\alpha}\left(g_\sigma^{\mu\nu}\frac{\partial H}{\partial g_\alpha^{\mu\nu}}\right)-\frac{\partial H}{\partial x^\sigma}$$

Or:

$$\frac{\partial}{\partial x^\alpha}\left(g_\sigma^{\mu\nu}\frac{\partial H}{\partial g_\alpha^{\mu\nu}}\right)-\frac{\partial}{\partial x^\sigma}(H)$$

$$\frac{\partial}{\partial x^\alpha}\left(g_\sigma^{\mu\nu}\frac{\partial H}{\partial g_\alpha^{\mu\nu}}\right)-\frac{\partial}{\partial x^\alpha}(\delta_\sigma^\alpha H)$$

$$\frac{\partial}{\partial x^\alpha}\left(g_\sigma^{\mu\nu}\frac{\partial H}{\partial g_\alpha^{\mu\nu}}-\delta_\sigma^\alpha H\right)$$

According to Einstein's equation (49):

$$-2\kappa t_\sigma^\alpha=g_\sigma^{\mu\nu}\frac{\partial H}{\partial g_\alpha^{\mu\nu}}-\delta_\sigma^\alpha H$$

Fill this in equation (1):

$$\frac{1}{2\kappa}\left(\frac{\partial g^{\mu\nu}}{\partial x^\sigma}\frac{\partial \Gamma_{\mu\nu}^\alpha}{\partial x^\alpha}+\frac{\partial g^{\mu\nu}}{\partial x^\sigma}\Gamma_{\mu\beta}^\alpha\Gamma_{\nu\alpha}^\beta\right)+\frac{1}{2}\frac{\partial g^{\mu\nu}}{\partial x^\sigma}T_{\mu\nu}=0$$

Becomes:

$$\frac{1}{2\kappa}\left\{\frac{\partial}{\partial x^\alpha}(-2\kappa t_\sigma^\alpha)\right\}+\frac{1}{2}\frac{\partial g^{\mu\nu}}{\partial x^\sigma}T_{\mu\nu}=0$$

q.e.d.

7.5 Question about equation in Einstein's original work (English version)

Question:

I am reattaching the PDF file of Einstein's article for reference (Einstein, *Relativity: The Special and General Theory*, 1916 – revised edition 1924; also in *The Collected Papers of Albert Einstein*, 1997).

At the bottom of page 191 of the English version, there are three terms separated by equal signs. I don't understand why the first term is equal to the second. Einstein refers to equation (60), but that doesn't help me. Can you explain why these two terms are equal?

Answer:

Let's first look at equation (60) in Einstein's original German article.

On page 812 of the German original, there appears to be an error in equation (60):

$$\frac{\partial F_{\varrho\sigma}}{\partial x^{\tau}} + \frac{\partial F_{\sigma\tau}}{\partial x^{\varrho}} + \frac{\partial F_{\tau\varrho}}{\partial x^{\varrho}} = 0$$

This is probably incorrect and should be:

$$\frac{\partial F_{\varrho\sigma}}{\partial x^{\tau}} + \frac{\partial F_{\sigma\tau}}{\partial x^{\varrho}} + \frac{\partial F_{\tau\varrho}}{\partial x^{\sigma}} = 0 \qquad (60)$$

In the English translation (page 189), it has already been corrected.

On page 191 (in English), we find the following equation:

$$F^{\mu\nu}\,\frac{\partial F_{\sigma\mu}}{\partial x^{\nu}} = -\frac{1}{2}F^{\mu\nu}\,\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} = -\frac{1}{2}g^{\mu\alpha}\,g^{\nu\beta}\,\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} \qquad (1)$$

We focus on the first equal sign:
why does

$$F^{\mu\nu}\,\frac{\partial F_{\sigma\mu}}{\partial x^{\nu}} = -\frac{1}{2}F^{\mu\nu}\,\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} \ \ ?$$

According to equation (60), the following applies:

$$\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} + \frac{\partial F_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial F_{\sigma\mu}}{\partial x^{\nu}} = 0$$

From this it follows:

$$\frac{\partial F_{\sigma\mu}}{\partial x^{\nu}} = -\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} - \frac{\partial F_{\nu\sigma}}{\partial x^{\mu}}$$

Filling in equation (1) yields:

$$F^{\mu\nu}\,\frac{\partial F_{\sigma\mu}}{\partial x^{\nu}} = F^{\mu\nu}\left(-\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} - \frac{\partial F_{\nu\sigma}}{\partial x^{\mu}}\right) = -F^{\mu\nu}\,\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} - F^{\mu\nu}\,\frac{\partial F_{\nu\sigma}}{\partial x^{\mu}}$$

Now we split each term into two equal parts:

$$= -\frac{1}{2}F^{\mu\nu}\,\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} - \frac{1}{2}F^{\mu\nu}\,\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} - \frac{1}{2}F^{\mu\nu}\,\frac{\partial F_{\nu\sigma}}{\partial x^{\mu}} - \frac{1}{2}F^{\mu\nu}\,\frac{\partial F_{\nu\sigma}}{\partial x^{\mu}}$$

$$= -\frac{1}{2}F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} - \frac{1}{2}\left(F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} + F^{\mu\nu}\frac{\partial F_{\nu\sigma}}{\partial x^{\mu}} + F^{\mu\nu}\frac{\partial F_{\nu\sigma}}{\partial x^{\mu}}\right)$$

Now we rearrange the indices (dummy indices may be swapped):

$$= -\frac{1}{2}F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} - \frac{1}{2}\left(F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} + F^{\mu\nu}\frac{\partial F_{\nu\sigma}}{\partial x^{\mu}} + F^{\nu\mu}\frac{\partial F_{\mu\sigma}}{\partial x^{\nu}}\right)$$

Because $F^{\nu\mu} = -F^{\mu\nu}$, the sign of the last term changes:

$$= -\frac{1}{2}F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} - \frac{1}{2}\left(F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} + F^{\mu\nu}\frac{\partial F_{\nu\sigma}}{\partial x^{\mu}} - F^{\mu\nu}\frac{\partial F_{\mu\sigma}}{\partial x^{\nu}}\right)$$

By swapping the indices of $\frac{\partial F_{\sigma\mu}}{\partial x^{\nu}}$ and changing the sign:

$$= -\frac{1}{2}F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} - \frac{1}{2}\left(F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} + F^{\mu\nu}\frac{\partial F_{\nu\sigma}}{\partial x^{\mu}} + F^{\mu\nu}\frac{\partial F_{\sigma\mu}}{\partial x^{\nu}}\right)$$

$$= -\frac{1}{2}F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} - \frac{1}{2}F^{\mu\nu}\left(\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} + \frac{\partial F_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial F_{\sigma\mu}}{\partial x^{\nu}}\right)$$

The expression in parentheses is exactly equation (60), which is equal to zero. So:

$$F^{\mu\nu}\frac{\partial F_{\sigma\mu}}{\partial x^{\nu}} = -\frac{1}{2}F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}}$$

Q.E.D.

7.6 Question about Einstein's equation (69)

Question:

Equation (69) in Einstein's book states:

$$k = \frac{8\pi K}{c^2} = 1.87.10^{-27} \qquad (E69)$$

Why does this number not correspond to the value we use today? And why is there still ac^2 in the denominator if Einstein previously stated that $c = 1$?

Answer:

In this book, Einstein worked with **CGS units** (centimeter–gram–second), whereas today we usually use **SI units** (meter–kilogram–second). This causes differences in the numerical values of natural constants such as K (the gravitational constant), depending on the unit system.

Step 1: Interpretation of Einstein's notation

- In Einstein's equation:

$$k = \frac{8\pi K}{c^2}$$

Is:

- K is the gravitational constant in CGS units:
$$K = 6.67 * 10^{-8}\ \ cm^3 g^{-1} s^{-2}$$
- $c = 3.00 * 10^{10}\ \text{cm/s}$

If we fill in these values:

$$k = \frac{8\pi(6.67 * 10^{-8})}{(3.00 * 10^{10})^2} \approx 1.87 * 10^{-27}$$

That is exactly the value Einstein mentions. His calculation is therefore correct within the CGS system.

Step 2: Conversion to modern units (SI)

In modern literature, we use the following for Einstein's field equations:

$$k = \frac{8\pi G}{c^4} \approx 2.07.10^{-43}$$

with:

- $G = 6.674 * 10^{-11}\ \text{m}^3\text{kg}^{-1}\text{s}^{-2}$
- $c = 3.00 * 10^8\ \text{m/s}$

Filled in:

$$k = \frac{8\pi(6.67 * 10^{-11})}{(3.00 * 10^8)^4} \approx 2.07.10^{-43}\ \text{m}^{-1}\text{kg}^{-1}\text{s}^2$$

Step 3: Why is there still a c^2 in Einstein's equation?

Although Einstein uses the convention c=1 (natural units) elsewhere in the book, he continues to explicitly note *c* here. This is probably because he wanted to give a **numerical estimate** at this point that corresponds to concrete physical units, and therefore temporarily disregarded the convention c=1.

Conclusion:

- Einstein's value of $k \approx 1.87 * 10^{-27}$ is correct in **CGS units**.
- In **modern SI units**, $k \approx 2.07.10^{-43}$.
- The notation with explicit c^2 indicates that Einstein **had not yet switched to natural units** (where c=1) in order to calculate a concrete value.

Appendices

Appendix 1 Formulas of General Relativity

Below we summarize a number of previously derived general relativity and Schwarzschild formulas. We then derive all formulas relevant for calculations in various chapters. In this appendix we use Einstein notation.

General Relativity Formulas:

Einstein's field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Where:

- $R_{\mu\nu}$ is the Ricci tensor
- $g_{\mu\nu}$ the metric tensor
- R the Ricci scalar
- λ the cosmological constant
- $T_{\mu\nu}$ the energy-momentum tensor

Schwarzschild metric (in spherical coordinates):

$$ds^2 = \left(1 - \frac{2GM}{\mathrm{r}c^2}\right) c^2 dt^2 - \left(1 - \frac{2GM}{\mathrm{r}c^2}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\emptyset^2$$

Where:

- ds^2 is the space-time interval
- G the gravitational constant
- M the mass of the central object
- r the radial coordinate
- θ and $\emptyset$ are spherical coordinates.

The coefficients are therefore independent of t and $\emptyset$, but only depend on r and θ !

Time dilation for a spherical object (Gravitational Time Dilation):

$$\Delta\tau = \Delta t \sqrt{1 - \frac{2GM}{rc^2}}$$

Where:

- $\Delta\tau$ the proper time is for an observer at a distance r
- Δt the time is for a distant observer

Path of light (null geodesics):

$$\mathrm{d}s^2 = \mathrm{o} => \left(1 - \frac{2GM}{\mathrm{r}c^2}\right) c^2 dt^2 = \left(1 - \frac{2GM}{\mathrm{r}c^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\emptyset^2$$

Radius of curvature of light around a mass:

The deviation of a light ray in the presence of a gravitational field is given by:

$$\delta\emptyset = \frac{4GM}{\mathrm{r}c^2}$$

Appendix 1.1 Summary and derivation of further relevant formulas:

In this section, we will derive the relevant formulas for the specific calculations in the chapters. This includes the derivation of the metric tensor, geodesic equations, and the energy-momentum tensor in different configurations.

$$dx^m = \frac{\partial x^m}{\partial y^r} dy^r$$
$$ds^2 = \eta_{mn} d\xi^m d\xi^n$$
$$ds^2 = g_{mn}(x) dx^m dx^n = g_{pq}(y) dy^p dy^q$$
$$g_{pq}(y) = g_{mn}(x) \frac{dx^m}{dy^p} \frac{dx^n}{dy^q}$$
$$V'^n(y) = \frac{\partial y^n}{\partial x^m} V^m(x)$$
$$W_p'(y) = \frac{\partial x^q}{\partial y^p} W_q'$$
$$T_{mn}(x) = \frac{\partial V^m(x)}{\partial x^n}$$
$$T_{mn}(y) = \frac{\partial x^r}{\partial y^m} \frac{\partial x^s}{\partial y^n} T_{rs}(x)$$
$$T^{mn}(y) = \frac{\partial y^m}{\partial x^r} \frac{\partial y^n}{\partial x^s} T^{rs}(x)$$
$$T^{rs}(x) = A_x^r B_x^s$$
$$E_\mu = g_{\mu\vartheta} E^\vartheta$$
$$E^\mu = g^{\mu\vartheta} E_\vartheta = g^{\mu\vartheta} g_{\vartheta\rho} E^\rho = \delta_\rho^\mu E^\rho = E^\mu$$

Line segment in small area applies: Pythagoras:

$$ds^2 = \delta_{mn} \frac{\partial x^m}{\partial y^n} dy^n . \frac{\partial x^n}{\partial y^s} dy^s$$

Transforming to another frame:

$$ds^2 = \boxed{\delta_{mn} \frac{\partial x^m}{\partial y^r} . \frac{\partial x^n}{\partial y^s}} dy^r dy^s$$

$$\boldsymbol{metric\ tensor}: \ g_{mn} = \delta_{mn} \frac{\partial x^m}{\partial y^r} . \frac{\partial x^n}{\partial y^s}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Geodesic equation:

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma_{\mu\nu}^\lambda . \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0 \qquad \Gamma_{\mu\nu}^\lambda \equiv \frac{\partial x^\lambda}{\partial \xi^\alpha} \ \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}$$
$$T_{\mu\vartheta}'(y) = \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\vartheta} T_{\alpha\beta}$$
$$T'^{\mu\vartheta}(y) = \frac{\partial y^\mu}{\partial x^\alpha} \frac{\partial y^\vartheta}{\partial x^\beta} T_{\alpha\beta}$$
$$T_\mu'^\vartheta(y) = \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial y^\vartheta}{\partial x^\beta} T_\alpha^\beta$$
$$g_{\mu\alpha} g^{\alpha\vartheta} = \delta_\mu^\vartheta$$

Contraction:

$$A^\mu = g^{\mu\vartheta} A_\vartheta$$
$$A_\mu = g_{\mu\vartheta} A^\vartheta$$
$$so: \ A.B = g_{\mu\vartheta} A^\mu B^\vartheta \equiv A_\vartheta B^\vartheta$$

Ricci Tensor:

$$\mathrm{R}_{\mu\nu} = \mathrm{R}_{\mu\rho\nu}^\rho = \Gamma_{\mu\nu,\rho}^\rho - \Gamma_{\rho\mu,\nu}^\rho + \Gamma_{\rho\lambda}^\rho \Gamma_{\nu\mu}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\rho\mu}^\lambda$$
$$G_{\mu\nu} = \Gamma_{\mu\nu,\rho}^\rho - \Gamma_{\nu\lambda}^\rho \Gamma_{\rho\mu}^\lambda \ only\ if\ g = \det(g_{\mu\nu}) = -1$$

Christoffel symbol:

$$\boxed{\Gamma_{\mu\nu}^\rho = \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right\}}$$

Ricci scalar:

$$g^{\mu\nu} \mathrm{R}_{\mu\nu} = R_\mu^\mu$$
$$R = g^{ab} (\Gamma_{ab,c}^c - \Gamma_{ac,b}^c + \Gamma_{ab}^d \Gamma_{cd}^c - \Gamma_{ac}^d \Gamma_{bd}^c)$$
$$R = 2g^{ab} (\Gamma_{a[b,c]}^c + \Gamma_{a[b}^d \Gamma_{c]d}^c)$$

Below, we perform a number of additional calculations for Schwarzschild geometries

Appendix 1.2 Schwarzschild Metric – Polar Coordinates

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta \, d\emptyset^2$$

$$\sigma^2 = 1 - \frac{R_s}{r} \quad hier\ is\!:\ R_s = \frac{2GM}{c^2}$$

$$g_{00} = g_{tt} \qquad g_{22} = g_{\theta\theta}$$
$$g_{11} = g_{rr} \qquad g_{33} = g_{\emptyset\emptyset}$$

Schwarzschild in polar coordinates (in plane $\theta = 90^0$)

$$g_{00} = \boldsymbol{\sigma^2} \qquad g^{00} = \frac{\mathbf{1}}{\boldsymbol{\sigma^2}}$$
$$g_{11} = \frac{\mathbf{-1}}{\boldsymbol{\sigma^2}} \qquad g^{11} = \boldsymbol{-\sigma^2}$$
$$g_{22} = \boldsymbol{-r^2} \qquad g^{22} = \frac{-1}{\boldsymbol{r^2}}$$
$$g_{33} = \boldsymbol{-r^2 \sin^2\theta} = -r^2 \qquad g^{33} = \frac{-1}{\boldsymbol{r^2 \sin^2\theta}} = \frac{-1}{r^2}$$
$$\frac{d\sigma}{dr} = \frac{\boldsymbol{R_s}}{\boldsymbol{2r^2\sigma}}$$

Metric first derivative for spherical coordinates

$$\frac{\partial g_{00}}{\partial r} = \frac{\boldsymbol{R_s}}{\boldsymbol{r^2}} \qquad \frac{\partial g_{11}}{\partial r} = \frac{\boldsymbol{R_s}}{\boldsymbol{r^2\sigma^4}}$$
$$\frac{\partial g_{22}}{\partial r} = \boldsymbol{-2r} \qquad \frac{\partial g_{33}}{\partial r} = \boldsymbol{-2r\sin^2\theta} = \boldsymbol{-2r}$$
$$\frac{\partial g_{33}}{\partial \theta} = \boldsymbol{-2r^2.sin(\theta)\cos(\theta)} = \mathbf{0}$$

Metric second derivative for spherical coordinates

$$\frac{\partial^2 g_{00}}{\partial r^2} = \frac{\boldsymbol{-2R_s}}{\boldsymbol{r^3}} \qquad \frac{\partial^2 g_{11}}{\partial r^2} = \frac{\boldsymbol{-2R_s}}{\boldsymbol{r^3\sigma^6}}$$
$$\frac{\partial^2 g_{22}}{\partial r^2} = \mathbf{-2} \qquad \frac{\partial^2 g_{33}}{\partial r^2} = \boldsymbol{-2\sin^2\theta} = \mathbf{-2}$$
$$\frac{\partial^2 g_{33}}{\partial\theta\partial r} = \boldsymbol{-4r.sin(\theta)\cos(\theta)} = \mathbf{0}$$
$$\frac{\partial^2 g_{33}}{\partial\theta^2} = \boldsymbol{2r^2.(\sin^2(\theta) - \cos^2(\theta))} = \boldsymbol{2r^2}$$

Schwarzschild polar coordinates:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right\}$$
$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2} g^{00} \left\{ \frac{\partial g_{00}}{\partial r} \right\} = \frac{\boldsymbol{R_s}}{\boldsymbol{2r^2\sigma^2}}$$
$$\Gamma^1_{00} = \frac{1}{2} g^{11} \left\{ -\frac{\partial g_{00}}{\partial r} \right\} = \frac{\boldsymbol{\sigma^2 R_s}}{\boldsymbol{2r^2}}$$
$$\Gamma^1_{11} = \frac{1}{2} g^{11} \left\{ \frac{\partial g_{11}}{\partial r} \right\} = \frac{\boldsymbol{-R_s}}{\boldsymbol{2r^2\sigma^2}}$$
$$\Gamma^1_{22} = \frac{1}{2} g^{11} \left\{ -\frac{\partial g_{22}}{\partial r} \right\} = \boldsymbol{-r\sigma^2}$$
$$\Gamma^1_{33} = \frac{1}{2} g^{11} \left\{ -\frac{\partial g_{33}}{\partial r} \right\} = \boldsymbol{-r\sigma^2\sin^2\theta} = \boldsymbol{-r\sigma^2}$$
$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2} g^{22} \left\{ \frac{\partial g_{22}}{\partial r} \right\} = \frac{\mathbf{1}}{\boldsymbol{r}}$$
$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2} g^{33} \left\{ \frac{\partial g_{33}}{\partial r} \right\} = \frac{\mathbf{1}}{\boldsymbol{r}}$$
$$\Gamma^2_{33} = \frac{1}{2} g^{22} \left\{ -\frac{\partial g_{33}}{\partial \theta} \right\} = \boldsymbol{-\cos\theta\sin\theta} = \mathbf{0}$$
$$\Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2} g^{33} \left\{ \frac{\partial g_{33}}{\partial \theta} \right\} = \frac{\boldsymbol{\cos\theta}}{\boldsymbol{\sin\theta}} = \mathbf{0}$$

In **r, theta, phi** coordinates:

$$\frac{\partial\Gamma^0_{01}}{\partial r} = \frac{\partial\Gamma^0_{10}}{\partial r} = \frac{\mathbf{R_s(R_s - 2r)}}{\mathbf{2r^4\sigma^4}}$$
$$\frac{\partial\Gamma^1_{00}}{\partial r} = \frac{\mathbf{R_s(3R_s - 2r)}}{\mathbf{2r^4}}$$
$$\frac{\partial\Gamma^1_{11}}{\partial r} = \frac{\mathbf{R_s(2r - Rs)}}{\mathbf{2r^4\sigma^4}}$$
$$\frac{\partial\Gamma^1_{22}}{\partial r} = \mathbf{-1}$$
$$\frac{\partial\Gamma^1_{33}}{\partial r} = \boldsymbol{-\sin^2\theta}$$
$$\frac{\partial\Gamma^2_{12}}{\partial r} = \frac{\partial\Gamma^2_{21}}{\partial r} = \frac{\partial\Gamma^3_{13}}{\partial r} = \frac{\partial\Gamma^3_{31}}{\partial r} = \frac{\mathbf{-1}}{\mathbf{r^2}}$$
$$\frac{\partial\Gamma^2_{33}}{\partial\theta} = \boldsymbol{-\cos^2\theta + \sin^2\theta} = \mathbf{1}$$
$$\frac{\partial\Gamma^3_{23}}{\partial\theta} = \frac{\partial\Gamma^3_{32}}{\partial\theta} = \frac{-1}{\boldsymbol{\sin^2\theta}} = \mathbf{-1}$$

Schwarzschild in r, theta, phi coordinates:

$$\Gamma_{\mu\nu}^{\rho} = \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\}$$

First derivative of the Christoffel symbol:

$$\frac{\partial \Gamma_{\mu\nu}^{\rho}}{\partial x^{\delta}} = \frac{1}{2} \frac{\partial g^{\rho\alpha}}{\partial x^{\delta}} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\} + \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial^2 g_{\nu\alpha}}{\partial x^{\mu} \partial x^{\delta}} + \frac{\partial^2 g_{\mu\alpha}}{\partial x^{\nu} \partial x^{\delta}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha} \partial x^{\delta}} \right\}$$

$$\frac{\partial \Gamma_{\mu\nu}^{\rho}}{\partial x^{\delta}} = \frac{-1}{2} (g^{\rho\alpha})^2 \frac{\partial g_{\rho\alpha}}{\partial x^{\delta}} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\} + \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial^2 g_{\nu\alpha}}{\partial x^{\mu} \partial x^{\delta}} + \frac{\partial^2 g_{\mu\alpha}}{\partial x^{\nu} \partial x^{\delta}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha} \partial x^{\delta}} \right\}$$

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Appendix 1.3 Schwarzschild Metric - x,y,z Coordinates

$$x_0 = t_\infty \qquad dx_0 = dt_\infty$$

$$x_1 = \frac{r^3}{3} \qquad dx_1 = r^2.dr \qquad \frac{\boldsymbol{dr}}{\boldsymbol{dx_1}} = \frac{\mathbf{1}}{\boldsymbol{r^2}}$$

$$x_2 = -\cos\theta = 0 \quad dx_2 = \sin\theta\,.d\theta = d\theta \quad \frac{\boldsymbol{d\theta}}{\boldsymbol{dx_2}} = \frac{\mathbf{1}}{\boldsymbol{\sin\theta}}$$

$$x_3 = \emptyset \qquad dx_3 = d\emptyset$$

$$ds^2 = \sigma^2 c^2 dt_\infty{}^2 - \frac{dx_1{}^2}{r^4\sigma^2} - \frac{r^2 dx_2{}^2}{\sin^2\theta} - r^2\sin^2\theta\, dx_3{}^2$$

Assumption at equator level $\theta = 90^0 \Rightarrow \sin\theta = 1$

$$\boxed{ds^2 = \sigma^2 c^2 dt_\infty{}^2 - \frac{dx_1{}^2}{r^4\sigma^2} - r^2 dx_2{}^2 - r^2 dx_3{}^2}$$

Schwarzschild metric in x, y, z

$$g_{00} = \sigma^2 \qquad g^{00} = \frac{1}{\sigma^2}$$

$$g_{11} = -\frac{1}{r^4\sigma^2} \qquad g^{11} = -r^4\sigma^2$$

$$g_{22} = -\frac{r^2}{\sin^2\theta} \qquad g^{22} = -\frac{\sin^2\theta}{r^2}$$

$$g_{33} = -r^2\sin^2\theta = -r^2 \qquad g^{33} = \frac{\mathbf{-1}}{r^2\sin^2\theta} = \frac{-1}{r^2}$$

g's depend on r (so x_1) and θ (so x_2):

$$\frac{dr}{dx_1} = \frac{\mathbf{1}}{\boldsymbol{r^2}} \qquad \frac{d\sigma}{dx_1} = \frac{\boldsymbol{R_s}}{\boldsymbol{2r^4\sigma}} \qquad \frac{d\theta}{dx_2} = \frac{\mathbf{1}}{\boldsymbol{\sin\theta}}$$

Metric derivative for x, y, z

$$\frac{\partial g_{00}}{\partial x_1} = \frac{\partial g_{00}}{\partial r}\frac{dr}{dx_1} = 2\sigma\frac{R_s}{2r^4\sigma} = \frac{\boldsymbol{R_s}}{\boldsymbol{r^4}}$$

$$\frac{\partial g_{11}}{\partial x_1} = \frac{\boldsymbol{4r - 3R_s}}{\boldsymbol{r^8\sigma^4}}$$

$$\frac{\partial g_{22}}{\partial x_1} = \frac{\partial g_{22}}{\partial r}\frac{dr}{dx_1} = r^{-2}\left(\frac{-2r}{\sin^2\theta}\right) = \frac{\mathbf{-2}}{\boldsymbol{r\sin^2\theta}} = \frac{\mathbf{-2}}{\boldsymbol{r}}$$

$$\frac{\partial g_{33}}{\partial x_1} = r^{-2}(-2r\sin^2\theta) = \frac{\boldsymbol{-2\sin^2\theta}}{\boldsymbol{r}} = \frac{\mathbf{-2}}{\boldsymbol{r}}$$

$$\frac{\partial g_{22}}{\partial x_2} = \frac{2r^2\cos(\theta)}{sin^3(\theta)}.\frac{1}{sin\,\theta} = \frac{\boldsymbol{2r^2\cos(\theta)}}{\boldsymbol{sin^4(\theta)}} = \mathbf{0}$$

$$\frac{\partial g_{33}}{\partial x_2} = \frac{\partial g_{33}}{\partial \theta}\frac{d\theta}{dx_2} = \left(\boldsymbol{-2r^2.sin(\theta)cos(\theta)}\right)\frac{1}{sin\,\theta}$$
$$= \boldsymbol{-2.r^2.cos(\theta) = 0}$$

Metric second derivative for x, y, z coordinates

$$\frac{\partial^2 g_{00}}{\partial x_1{}^2} = \frac{\boldsymbol{-4R_s}}{\boldsymbol{r^7}} \qquad \frac{\partial^2 g_{11}}{\partial x_1{}^2} = \frac{\boldsymbol{-2\left(14r^2 + 9R_s{}^2 - 22rR_s\right)}}{\boldsymbol{r^{12}\sigma^6}}$$

$$\frac{\partial^2 g_{22}}{\partial x_1{}^2} = \frac{\mathbf{2}}{\boldsymbol{r^4 sin^2(\theta)}} = \frac{\mathbf{2}}{\boldsymbol{r^4}}$$

$$\frac{\partial^2 g_{22}}{\partial x_2{}^2} = \frac{\boldsymbol{-2r^2\left(1 + 3\cos^2(\theta)\right)}}{\boldsymbol{sin^6(\theta)}} = \boldsymbol{-2r^2}$$

$$\frac{\partial^2 g_{22}}{\partial x_1\partial x_2} = \frac{\partial^2 g_{22}}{\partial x_2\partial x_1} = \frac{\boldsymbol{4cos(\theta)}}{\boldsymbol{r sin^4(\theta)}} = \mathbf{0}$$

$$\frac{\partial^2 g_{33}}{\partial x_1{}^2} = \frac{\boldsymbol{2sin^2(\theta)}}{\boldsymbol{r^4}} = \frac{\mathbf{2}}{\boldsymbol{r^4}}$$

$$\frac{\partial^2 g_{33}}{\partial x_1\partial x_2} = \frac{\partial^2 g_{33}}{\partial x_2\partial x_1} = \frac{\boldsymbol{-4cos(\theta)}}{\boldsymbol{r}} = \mathbf{0}$$

$$\frac{\partial^2 g_{33}}{\partial x_2{}^2} = \boldsymbol{2r^2.\sin\theta}\frac{\mathbf{1}}{\boldsymbol{\sin\theta}} = \boldsymbol{2r^2}$$

Schwarzschild in x, y, z

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right\}$$

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2}g^{00}\left\{\frac{\partial g_{00}}{\partial x^1}\right\} = \frac{1}{2}\frac{1}{\sigma^2}\frac{R_s}{r^4} = \frac{\boldsymbol{R_s}}{\boldsymbol{2r^4\sigma^2}}$$

$$\Gamma^1_{00} = \frac{1}{2}g^{11}\left\{-\frac{\partial g_{00}}{\partial x^1}\right\} = \frac{1}{2}(-r^4\sigma^2)\frac{-R_s}{r^4} = \frac{\boldsymbol{R_s\sigma^2}}{\mathbf{2}}$$

$$\Gamma^1_{11} = \frac{1}{2}g^{11}\left\{\frac{\partial g_{11}}{\partial x^1}\right\} = \frac{1}{2}(-r^4\sigma^2)\frac{4r - 3R_s}{r^8\sigma^4} = \frac{\boldsymbol{3R_s - 4r}}{\boldsymbol{2r^4\sigma^2}}$$

$$\Gamma^1_{22} = \frac{1}{2}g^{11}\left\{-\frac{\partial g_{22}}{\partial x^1}\right\} = \frac{1}{2}(-r^4\sigma^2)\frac{2}{r\sin^2\theta} = \frac{\boldsymbol{-r^3\sigma^2}}{\boldsymbol{\sin^2\theta}}$$
$$= \boldsymbol{-r^3\sigma^2}$$

$$\Gamma^1_{33} = \frac{1}{2}g^{11}\left\{-\frac{\partial g_{33}}{\partial x^1}\right\} = \frac{1}{2}(-r^4\sigma^2)\frac{2\sin^2\theta}{r}$$
$$= \boldsymbol{-r^3\sigma^2\sin^2\theta = -r^3\sigma^2}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}g^{22}\left\{\frac{\partial g_{22}}{\partial x^1}\right\} = \frac{1}{2}(-\frac{\sin^2\theta}{r^2})\frac{-2}{r\sin^2\theta} = \frac{\mathbf{1}}{\boldsymbol{r^3}}$$

$$\Gamma^2_{33} = \frac{1}{2}g^{22}\left\{-\frac{\partial g_{33}}{\partial x^2}\right\} = \frac{1}{2}\left(-\frac{\sin^2\theta}{r^2}\right)(2.r^2.\cos(\theta))$$
$$= \boldsymbol{-\sin^2\theta\cos\theta} = 0$$

$$\Gamma^2_{22} = \frac{1}{2}g^{22}\left\{\frac{\partial g_{22}}{\partial x^2}\right\} = \frac{1}{2}\left(-\frac{\sin^2\theta}{r^2}\right)\frac{2r^2\cos(\theta)}{sin^4(\theta)}$$
$$= \frac{\boldsymbol{-cos(\theta)}}{\boldsymbol{\sin^2(\theta)}} = \mathbf{0}$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2}g^{33}\left\{\frac{\partial g_{33}}{\partial x^1}\right\} = \frac{1}{2}(\frac{-1}{r^2\sin^2\theta})\frac{-2\sin^2\theta}{r}$$
$$= \frac{\mathbf{1}}{\boldsymbol{r^3}}$$

$$\Gamma_{23}^{3} = \Gamma_{32}^{3} = \frac{1}{2} g^{33} \left\{ \frac{\partial g_{33}}{\partial x^2} \right\}$$
$$= \frac{1}{2} \left(\frac{-1}{r^2 \sin^2 \theta} \right) (-2.r^2.\cos(\theta))$$
$$= \frac{\mathbf{\cos\theta}}{\mathbf{\sin^2(\theta)}} = \mathbf{0}$$

For x, y, z coordinates:

$$\frac{\partial\Gamma_{01}^{0}}{\partial x_1} = \frac{\partial\Gamma_{10}^{0}}{\partial x_1} = \frac{\mathbf{R_s(3R_s - 4r)}}{\mathbf{2r^8\sigma^4}}$$
$$\frac{\partial\Gamma_{00}^{1}}{\partial x_1} = \frac{\mathbf{R_s}^{\mathbf{2}}}{\mathbf{2r^4}}$$
$$\frac{\partial\Gamma_{11}^{1}}{\partial x_1} = \frac{\mathbf{6}}{\mathbf{r^6\sigma^4}} - \frac{\mathbf{10R_s}}{\mathbf{r^7\sigma^4}} + \frac{\mathbf{4.5R_s}^{\mathbf{2}}}{\mathbf{r^8\sigma^4}}$$
$$\frac{\partial\Gamma_{22}^{1}}{\partial x_1} = \frac{\mathbf{2R_s - 3r}}{\mathbf{r\sin^2\theta}} = \mathbf{-3} + \frac{\mathbf{2R_s}}{\mathbf{r}}$$
$$\frac{\partial\Gamma_{33}^{1}}{\partial x_1} = \left(\mathbf{-3} + \frac{\mathbf{2R_s}}{\mathbf{r}}\right).\mathbf{\sin^2\theta} \;\; = \mathbf{-3} + \frac{\mathbf{2R_s}}{\mathbf{r}}$$
$$\frac{\partial\Gamma_{12}^{2}}{\partial x_1} = \frac{\partial\Gamma_{21}^{2}}{\partial x_1} = \frac{\partial\Gamma_{13}^{3}}{\partial x_1} = \frac{\partial\Gamma_{31}^{3}}{\partial x_1} = \frac{\mathbf{-3}}{\mathbf{r^6}}$$
$$\frac{\partial\Gamma_{33}^{2}}{\partial x_1} = \frac{\partial\Gamma_{22}^{2}}{\partial x_1} = \frac{\partial\Gamma_{23}^{3}}{\partial x_1} = \frac{\partial\Gamma_{32}^{3}}{\partial x_1} = \mathbf{0}$$
$$\frac{\partial\Gamma_{22}^{1}}{\partial x_2} = \frac{\mathbf{2r^3\sigma^2\cos\theta}}{\mathbf{\sin^4\theta}} = \mathbf{0}$$
$$\frac{\partial\Gamma_{33}^{1}}{\partial x_2} = \mathbf{-2r^3\sigma^2\cos\theta = 0}$$
$$\frac{\partial\Gamma_{33}^{2}}{\partial x_2} = -3\,\mathbf{\cos^2\theta + 1 = 1}$$
$$\frac{\partial\Gamma_{22}^{2}}{\partial x_2} = \frac{\mathbf{1 + \cos^2\theta}}{\mathbf{\sin^4\theta}} = \mathbf{1}$$
$$\frac{\partial\Gamma_{23}^{3}}{\partial x_2} = \frac{\partial\Gamma_{32}^{3}}{\partial x_2} = \frac{\mathbf{-1 - \cos^2\theta}}{\mathbf{\sin^4\theta}} = \mathbf{-1}$$

$$R^i_{jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^u_{jl}\,\Gamma^i_{uk} - \Gamma^u_{jk}\,\Gamma^i_{ul}$$

$$\mathbf{R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} = \Gamma^{\rho}_{\mu\nu,\rho} - \Gamma^{\rho}_{\mu\rho,\nu} + \Gamma^{\lambda}_{\mu\nu}\,\Gamma^{\rho}_{\lambda\rho} - \Gamma^{\lambda}_{\mu\rho}\,\Gamma^{\rho}_{\lambda\nu}}$$

$$R_{\mu\nu} = R^{\rho}_{\mu\nu\rho} = -\Gamma^{\rho}_{\mu\nu,\rho} + \Gamma^{\rho}_{\mu\rho,\nu} - \Gamma^{\lambda}_{\mu\nu}\,\Gamma^{\rho}_{\lambda\rho} + \Gamma^{\lambda}_{\mu\rho}\,\Gamma^{\rho}_{\lambda\nu}$$

After some calculations, the conclusion was that, in order to get all elements of the Ricci tensor to zero in vacuum, the formula for the Christoffel symbol must start with a positive +1/2:

$$\Gamma^{\rho}_{\mu\nu} = +\frac{1}{2}g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right\}$$

The sign at the beginning of the Christoffel symbols does not affect the **product** of the Christoffel symbols in the element of the Ricci tensor, but only the sign of the first two terms: the **derivatives of the Christoffel symbols.**

Schwarzschild symmetry:

$$\begin{aligned} R_{\mu\nu} = {} & \Gamma^0_{\mu\nu,0} - \Gamma^0_{0\mu,\nu} + \Gamma^0_{0\lambda}\,\Gamma^{\lambda}_{\nu\mu} - \Gamma^0_{\nu\lambda}\,\Gamma^{\lambda}_{0\mu} \\ & +\Gamma^1_{\mu\nu,1} - \Gamma^1_{1\mu,\nu} + \Gamma^1_{1\lambda}\,\Gamma^{\lambda}_{\nu\mu} - \Gamma^1_{\nu\lambda}\,\Gamma^{\lambda}_{1\mu} \\ & +\Gamma^2_{\mu\nu,2} - \Gamma^2_{2\mu,\nu} + \Gamma^2_{2\lambda}\,\Gamma^{\lambda}_{\nu\mu} - \Gamma^2_{\nu\lambda}\,\Gamma^{\lambda}_{2\mu} \\ & +\Gamma^3_{\mu\nu,3} - \Gamma^3_{3\mu,\nu} + \Gamma^3_{3\lambda}\,\Gamma^{\lambda}_{\nu\mu} - \Gamma^3_{\nu\lambda}\,\Gamma^{\lambda}_{3\mu} \end{aligned}$$

$$R_{\mu\nu} = \Gamma^{\rho}_{\mu\nu,\rho} - \Gamma^{\rho}_{\rho\mu,\nu} + \Gamma^{\rho}_{\rho\lambda}\,\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\,\Gamma^{\lambda}_{\rho\mu}$$

$$R_{00} = \Gamma^1_{00,1} + \Gamma^1_{11}\,\Gamma^1_{00} + \Gamma^2_{21}\,\Gamma^1_{00} + \Gamma^3_{31}\,\Gamma^1_{00} - \Gamma^1_{00}\,\Gamma^0_{10} = \frac{{R_s}^2}{2r^4} - \frac{1}{2}\frac{4r-3R_s}{r^4\sigma^2}\frac{1}{2}R_s\sigma^2 - \frac{1}{2}R_s\sigma^2\frac{1}{2}\frac{R_s}{r^4\sigma^2} - \frac{1}{2}\frac{R_s}{r^4\sigma^2}\frac{1}{2}R_s\sigma^2$$

$$R_{11} = -\Gamma^0_{01,1} - \Gamma^2_{21,1} - \Gamma^3_{31,1} + \Gamma^0_{01}\,\Gamma^1_{11} + \Gamma^2_{21}\,\Gamma^1_{11} + \Gamma^3_{31}\,\Gamma^1_{11} - \Gamma^0_{10}\,\Gamma^0_{01} - \Gamma^2_{12}\,\Gamma^2_{21} - \Gamma^3_{13}\,\Gamma^3_{31}$$

$$R_{22} = \Gamma^1_{22,1} - \Gamma^3_{32,2} + \Gamma^0_{01}\,\Gamma^1_{22} + \Gamma^1_{11}\,\Gamma^1_{22} + \Gamma^2_{21}\,\Gamma^1_{22} + \Gamma^3_{31}\,\Gamma^1_{22} - \Gamma^1_{22}\,\Gamma^2_{12} - \Gamma^2_{21}\,\Gamma^1_{22}$$

$$R_{33} = \Gamma^1_{33,1} + \Gamma^0_{01}\,\Gamma^1_{33} + \Gamma^1_{11}\,\Gamma^1_{33} + \Gamma^2_{21}\,\Gamma^1_{33} - \Gamma^1_{33}\,\Gamma^3_{13}$$

For spherical coordinates and the Schwarzschild configuration with $\theta = 90^0$, the following elements of the Ricci tensor are relevant:

$$R_{00} = \mathbf{\Gamma^1_{00,1}} + \Gamma^1_{00}\,\Gamma^1_{11} + \Gamma^1_{00}\,\Gamma^2_{12} + \Gamma^1_{00}\,\Gamma^3_{13} - \Gamma^0_{01}\,\Gamma^1_{00}$$

$$R_{11} = -\mathbf{\Gamma^0_{10,1}} - \mathbf{\Gamma^2_{12,1}} - \mathbf{\Gamma^3_{13,1}} + \Gamma^1_{11}\Gamma^0_{10} + \Gamma^1_{11}\,\Gamma^2_{12} + \Gamma^1_{11}\,\Gamma^3_{13} - \Gamma^0_{10}\Gamma^0_{01} - \Gamma^2_{12}\,\Gamma^2_{21} - \Gamma^3_{13}\,\Gamma^3_{31}$$

$$R_{22} = \mathbf{\Gamma^1_{22,1}} - \mathbf{\Gamma^3_{23,2}} + \Gamma^1_{22}\,\Gamma^0_{10} + \Gamma^1_{22}\,\Gamma^1_{11} + \Gamma^1_{22}\,\Gamma^3_{13} + \Gamma^2_{22}\,\Gamma^3_{32} - \Gamma^2_{21}\,\Gamma^1_{22} - \Gamma^3_{23}\,\Gamma^3_{32}$$

$$R_{33} = +\mathbf{\Gamma^1_{33,1}} + \mathbf{\Gamma^2_{33,2}} + \Gamma^1_{33}\,\Gamma^0_{10} + \Gamma^1_{33}\,\Gamma^1_{11} + \Gamma^1_{33}\,\Gamma^2_{12} + \Gamma^2_{33}\,\Gamma^2_{22} - \Gamma^3_{31}\,\Gamma^1_{33} - \Gamma^3_{32}\,\Gamma^2_{33}$$

$$R_{33} = \sin^2\theta\,.\,R_{22}$$

When$\theta \neq 90^0$, then forR_{22} andR_{33} , respectively, there is an additional term $+\Gamma^2_{22}\,\Gamma^3_{32}$ en $+\Gamma^2_{33}\,\Gamma^2_{22}$.

Appendix 2 Derivation of the Christoffel Symbols in a General Form

It is demonstrated how the Christoffel symbol depends only on the elements of the metric tensor and its derivatives. This is useful when using it in a spreadsheet or program.

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\}$$

$$\frac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\gamma}} = \frac{1}{2} \frac{\partial g^{\rho\alpha}}{\partial x^{\gamma}} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\} + \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial^2 g_{\nu\alpha}}{\partial x^{\mu} \partial x^{\gamma}} + \frac{\partial^2 g_{\mu\alpha}}{\partial x^{\nu} \partial x^{\gamma}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha} \partial x^{\gamma}} \right\}$$

$$\frac{\partial g^{\rho\alpha}}{\partial x^{\gamma}} = \frac{\partial \frac{1}{g_{\rho\alpha}}}{\partial x^{\gamma}} = \frac{-1}{{g_{\rho\alpha}}^2} . \frac{\partial g_{\rho\alpha}}{\partial x^{\gamma}} = -(g^{\rho\alpha})^2 . \frac{\partial g_{\rho\alpha}}{\partial x^{\gamma}}$$

$$\frac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\gamma}} = \frac{-1}{2} (g^{\rho\alpha})^2 . \frac{\partial g_{\rho\alpha}}{\partial x^{\gamma}} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\} + \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial^2 g_{\nu\alpha}}{\partial x^{\mu} \partial x^{\gamma}} + \frac{\partial^2 g_{\mu\alpha}}{\partial x^{\nu} \partial x^{\gamma}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha} \partial x^{\gamma}} \right\}$$

$$\boldsymbol{\frac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\gamma}} = \frac{1}{2} g^{\rho\alpha} \left[-g^{\rho\alpha} . \frac{\partial g_{\rho\alpha}}{\partial x^{\gamma}} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\} + \left\{ \frac{\partial^2 g_{\nu\alpha}}{\partial x^{\mu} \partial x^{\gamma}} + \frac{\partial^2 g_{\mu\alpha}}{\partial x^{\nu} \partial x^{\gamma}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha} \partial x^{\gamma}} \right\} \right]}$$

Or:

$$\frac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\gamma}} = -g^{\rho\alpha} \cdot \frac{\partial g_{\rho\alpha}}{\partial x^{\gamma}} \cdot \Gamma^{\rho}_{\mu\nu} + \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial^2 g_{\nu\alpha}}{\partial x^{\mu} \partial x^{\gamma}} + \frac{\partial^2 g_{\mu\alpha}}{\partial x^{\nu} \partial x^{\gamma}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha} \partial x^{\gamma}} \right\}$$

Appendix 3 Mathematical Derivation of Schwarzschild

Here we will work out the Christoffel symbols for the metric tensor of the Schwarzschild configuration.

Schwarzschild in r, theta, phi coordinates:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\}$$

$\Gamma^{0}_{01} = \Gamma^{0}_{10} = \frac{1}{2} g^{00} \left\{ \frac{\partial g_{00}}{\partial r} \right\}$ $\Gamma^{1}_{00} = \frac{1}{2} g^{11} \left\{ -\frac{\partial g_{00}}{\partial r} \right\}$ $\Gamma^{1}_{11} = \frac{1}{2} g^{11} \left\{ \frac{\partial g_{11}}{\partial r} \right\}$ $\Gamma^{1}_{22} = \frac{1}{2} g^{11} \left\{ -\frac{\partial g_{22}}{\partial r} \right\}$

$\Gamma^{1}_{33} = \frac{1}{2} g^{11} \left\{ -\frac{\partial g_{33}}{\partial r} \right\}$ $\Gamma^{2}_{12} = \Gamma^{2}_{21} = \frac{1}{2} g^{22} \left\{ \frac{\partial g_{22}}{\partial r} \right\}$ $\Gamma^{3}_{13} = \Gamma^{3}_{31} = \frac{1}{2} g^{33} \left\{ \frac{\partial g_{33}}{\partial r} \right\}$ $\Gamma^{2}_{33} = \frac{1}{2} g^{22} \left\{ -\frac{\partial g_{33}}{\partial \theta} \right\}$

$\Gamma^{3}_{23} = \Gamma^{3}_{32} = \frac{1}{2} g^{33} \left\{ \frac{\partial g_{33}}{\partial \theta} \right\}$

All elements in the metric tensor are zero, except for the elements in the diagonal. This means that the contravariant elements are the direct inverse of the covariant components. So, for example $g^{00} = \frac{1}{g_{00}}$, and so on.

For **r, theta, phi** coordinates:

Derivatives of gamma with respect to x_1=r:

$$\mathbf{0011 = 0101} = \frac{\boldsymbol{\partial\Gamma^0_{01}}}{\boldsymbol{\partial r}} = \frac{\boldsymbol{\partial\Gamma^0_{10}}}{\boldsymbol{\partial r}} = \frac{\mathbf{1}}{\mathbf{2}}\left\{\frac{-1}{{g_{00}}^2}\left(\frac{\partial g_{00}}{\partial r}\right)^2 + \frac{1}{g_{00}}\frac{\partial^2 g_{00}}{\partial r^2}\right\} = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{g_{00}}}\left\{\frac{\mathbf{-1}}{\boldsymbol{g_{00}}}(\boldsymbol{g_{00}}')^2 + \boldsymbol{g_{00}}''\right\}$$

$$\mathbf{1001} = \frac{\boldsymbol{\partial\Gamma^1_{00}}}{\boldsymbol{\partial r}} = \frac{-1}{2}\left\{\frac{-1}{{g_{11}}^2}\frac{\partial g_{11}}{\partial r}\frac{\partial g_{00}}{\partial r} + \frac{1}{g_{11}}\frac{\partial^2 g_{00}}{\partial r^2}\right\} = \frac{\mathbf{-1}}{\mathbf{2}\boldsymbol{g_{11}}}\left\{\frac{\mathbf{-1}}{\boldsymbol{g_{11}}}\boldsymbol{g_{11}}'\boldsymbol{g_{00}}' + \boldsymbol{g_{00}}''\right\}$$

$$\mathbf{1111} = \frac{\boldsymbol{\partial\Gamma^1_{11}}}{\boldsymbol{\partial r}} = \frac{\mathbf{1}}{\mathbf{2}}\left\{\frac{-1}{{g_{11}}^2}\left(\frac{\partial g_{11}}{\partial r}\right)^2 + \frac{1}{g_{11}}\frac{\partial^2 g_{11}}{\partial r^2}\right\} = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{g_{11}}}\left\{\frac{\mathbf{-1}}{\boldsymbol{g_{11}}}(\boldsymbol{g_{11}}')^2 + \boldsymbol{g_{11}}''\right\}$$

$$\mathbf{1221} = \frac{\boldsymbol{\partial\Gamma^1_{22}}}{\boldsymbol{\partial r}} = \frac{-1}{2}\left\{\frac{-1}{{g_{11}}^2}\frac{\partial g_{11}}{\partial r}\frac{\partial g_{22}}{\partial r} + \frac{1}{g_{11}}\frac{\partial^2 g_{22}}{\partial r^2}\right\} = \frac{\mathbf{-1}}{\mathbf{2}\boldsymbol{g_{11}}}\left\{\frac{\mathbf{-1}}{\boldsymbol{g_{11}}}\boldsymbol{g_{11}}'\boldsymbol{g_{22}}' + \boldsymbol{g_{22}}''\right\}$$

$$\mathbf{1331} = \frac{\boldsymbol{\partial\Gamma^1_{33}}}{\boldsymbol{\partial r}} = \frac{-1}{2}\left\{\frac{-1}{{g_{11}}^2}\frac{\partial g_{11}}{\partial r}\frac{\partial g_{33}}{\partial r} + \frac{1}{g_{11}}\frac{\partial^2 g_{33}}{\partial r^2}\right\} = \frac{\mathbf{-1}}{\mathbf{2}\boldsymbol{g_{11}}}\left\{\frac{\mathbf{-1}}{\boldsymbol{g_{11}}}\boldsymbol{g_{11}}'\boldsymbol{g_{33}}' + \boldsymbol{g_{33}}''\right\}$$

$$\mathbf{2121 = 2211} = \frac{\boldsymbol{\partial\Gamma^2_{12}}}{\boldsymbol{\partial r}} = \frac{\boldsymbol{\partial\Gamma^2_{21}}}{\boldsymbol{\partial r}} = \frac{1}{2}\left\{\frac{-1}{{g_{22}}^2}\left(\frac{\partial g_{22}}{\partial r}\right)^2 + \frac{1}{g_{22}}\frac{\partial^2 g_{22}}{\partial r^2}\right\} = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{g_{22}}}\left\{\frac{\mathbf{-1}}{\boldsymbol{g_{22}}}(\boldsymbol{g_{22}}')^2 + \boldsymbol{g_{22}}''\right\}$$

$$\mathbf{3131 = 3311} = \frac{\boldsymbol{\partial\Gamma^3_{13}}}{\boldsymbol{\partial r}} = \frac{\boldsymbol{\partial\Gamma^3_{31}}}{\boldsymbol{\partial r}} = \frac{1}{2}\left\{\frac{-1}{{g_{33}}^2}\left(\frac{\partial g_{33}}{\partial r}\right)^2 + \frac{1}{g_{33}}\frac{\partial^2 g_{33}}{\partial r^2}\right\} = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{g_{33}}}\left\{\frac{\mathbf{-1}}{\boldsymbol{g_{33}}}(\boldsymbol{g_{33}}')^2 + \boldsymbol{g_{33}}''\right\}$$

$$\mathbf{2331} = \frac{\boldsymbol{\partial\Gamma^2_{33}}}{\boldsymbol{\partial r}} = \frac{-1}{2}\left\{\frac{-1}{{g_{22}}^2}\frac{\partial g_{22}}{\partial r}\frac{\partial g_{33}}{\partial \theta} + \frac{1}{g_{22}}\frac{\partial^2 g_{33}}{\partial r\partial\theta}\right\} = \frac{\mathbf{-1}}{\mathbf{2}\boldsymbol{g_{22}}}\left\{\frac{\mathbf{-1}}{\boldsymbol{g_{22}}}\boldsymbol{g_{22}}'\frac{\boldsymbol{\partial g_{33}}}{\boldsymbol{\partial\theta}} + \frac{\boldsymbol{\partial^2 g_{33}}}{\boldsymbol{\partial r\partial\theta}}\right\}$$

$$\mathbf{3231 = 3321} = \frac{\boldsymbol{\partial\Gamma^3_{23}}}{\boldsymbol{\partial r}} = \frac{\boldsymbol{\partial\Gamma^3_{32}}}{\boldsymbol{\partial r}} = \frac{1}{2}\left\{\frac{-1}{{g_{33}}^2}\frac{\partial g_{33}}{\partial r}\frac{\partial g_{33}}{\partial \theta} + \frac{1}{g_{33}}\frac{\partial^2 g_{33}}{\partial r\partial\theta}\right\} = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{g_{33}}}\left\{\frac{\mathbf{-1}}{\boldsymbol{g_{33}}}\boldsymbol{g_{33}}'\frac{\boldsymbol{\partial g_{33}}}{\boldsymbol{\partial\theta}} + \frac{\boldsymbol{\partial^2 g_{33}}}{\boldsymbol{\partial r\partial\theta}}\right\}$$

Derivatives of gamma with respect to $x_2 = \theta$:

$$\mathbf{1222} = \frac{\boldsymbol{\partial\Gamma^1_{22}}}{\partial\theta} = \frac{\mathbf{-1}}{\mathbf{2}\boldsymbol{g_{11}}}\frac{\boldsymbol{\partial^2 g_{22}}}{\boldsymbol{\partial r\partial\theta}}$$

$$\mathbf{1332} = \frac{\boldsymbol{\partial\Gamma^1_{33}}}{\partial\theta} = \frac{\mathbf{-1}}{\mathbf{2}\boldsymbol{g_{11}}}\frac{\boldsymbol{\partial^2 g_{33}}}{\boldsymbol{\partial r\partial\theta}}$$

$$\mathbf{2332} = \frac{\boldsymbol{\partial\Gamma^2_{33}}}{\boldsymbol{\partial\theta}} = \frac{-1}{2}\left\{\frac{-1}{{g_{22}}^2}\frac{\partial g_{22}}{\partial \theta}\frac{\partial g_{33}}{\partial \theta} + \frac{1}{g_{22}}\frac{\partial^2 g_{33}}{\partial \theta^2}\right\} = \frac{\mathbf{-1}}{\mathbf{2}\boldsymbol{g_{22}}}\left\{\frac{\mathbf{-1}}{\boldsymbol{g_{22}}}\frac{\boldsymbol{\partial g_{22}}}{\boldsymbol{\partial\theta}}\frac{\boldsymbol{\partial g_{33}}}{\boldsymbol{\partial\theta}} + \frac{\boldsymbol{\partial^2 g_{33}}}{\boldsymbol{\partial\theta^2}}\right\}$$

$$\mathbf{2222} = \frac{\boldsymbol{\partial\Gamma^2_{22}}}{\boldsymbol{\partial\theta}} = \frac{1}{2}\left\{\frac{-1}{{g_{22}}^2}\left(\frac{\partial g_{22}}{\partial \theta}\right)^2 + \frac{1}{g_{22}}\frac{\partial^2 g_{22}}{\partial \theta^2}\right\} = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{g_{22}}}\left\{\frac{\mathbf{-1}}{\boldsymbol{g_{22}}}\left(\frac{\boldsymbol{\partial g_{22}}}{\boldsymbol{\partial\theta}}\right)^2 + \frac{\boldsymbol{\partial^2 g_{22}}}{\boldsymbol{\partial\theta^2}}\right\}$$

$$\mathbf{3312 = 3132} = \frac{\boldsymbol{\partial\Gamma^3_{31}}}{\boldsymbol{\partial\theta}} = \frac{\boldsymbol{\partial\Gamma^3_{13}}}{\boldsymbol{\partial\theta}} = \frac{1}{2}\left\{\frac{-1}{{g_{33}}^2}\frac{\partial g_{33}}{\partial r}\frac{\partial g_{33}}{\partial \theta} + \frac{1}{g_{33}}\frac{\partial^2 g_{33}}{\partial r\partial\theta}\right\} = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{g_{33}}}\left\{\frac{\mathbf{-1}}{\boldsymbol{g_{33}}}\boldsymbol{g_{33}}'\frac{\boldsymbol{\partial g_{33}}}{\boldsymbol{\partial\theta}} + \frac{\boldsymbol{\partial^2 g_{33}}}{\boldsymbol{\partial r\partial\theta}}\right\}$$

$$\mathbf{3232 = 3322} = \frac{\boldsymbol{\partial\Gamma^3_{23}}}{\boldsymbol{\partial\theta}} = \frac{\boldsymbol{\partial\Gamma^3_{32}}}{\boldsymbol{\partial\theta}} = \frac{1}{2}\left\{\frac{-1}{{g_{33}}^2}\left(\frac{\partial g_{33}}{\partial \theta}\right)^2 + \frac{1}{g_{33}}\frac{\partial^2 g_{33}}{\partial \theta^2}\right\} = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{g_{33}}}\left\{\frac{\mathbf{-1}}{\boldsymbol{g_{33}}}\left(\frac{\boldsymbol{\partial g_{33}}}{\boldsymbol{\partial\theta}}\right)^2 + \frac{\boldsymbol{\partial^2 g_{33}}}{\boldsymbol{\partial\theta^2}}\right\}$$

Appendix 4 The Schwarzschild Formula Extended for Elektric Charges

The Reissner–Nordström-Metric

The correct solution within general relativity for a charged, non-rotating, spherically symmetric mass is the **Reissner–Nordström metric** (1918). This metric describes the spacetime interval around a charged mass and incorporates both gravitational and electric contributions:

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{4\pi\varepsilon_0 c^4 \mathrm{r}^2}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{4\pi\varepsilon_0 c^4 \mathrm{r}^2}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\emptyset^2$$

or equivalently:

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{r_s}{r} + \frac{\mathrm{r}_\mathrm{Q}^2}{\mathrm{r}^2}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r} + \frac{\mathrm{r}_\mathrm{Q}^2}{\mathrm{r}^2}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\emptyset^2)$$

where:

- $r_s = \frac{2GM}{c^2}$: Schwarzschild radius, gravitational term,
- $\mathrm{r}_\mathrm{Q}^2 = \frac{GQ^2}{4\pi\varepsilon_0 c^4}$: charge term,
- Q : elektric charge of the central object,
- M : mass of the object,
- G: gravitational constant,
- c: speed of light.

Interpretatie

- The first term $\left(\frac{r_s}{r}\right)$, represents the gravitational curvature of spacetime as in the Schwarzschild solution.
- The additional term $\frac{r_Q^2}{r^2}$ accounts for the repulsive (for like charges) electromagnetic effect according to general relativity.
- The metric reduces to the ordinary Schwarzschild solution when $Q = 0$ (i.e., no charge).
- For rotating or more complex charged objects (such as electrons), more general solutions exist, such as the **Kerr–Newman metric**.

Classical versus Relativistic Vacuum

In classical Newtonian physics, gravity is described by a gravitational field in vacuum.
However, according to Einstein, there is no gravitational “field” per se—spacetime itself is curved by the presence of mass-energy. In the Schwarzschild case, this means $T_{\mu\nu} = 0$.

In the Reissner–Nordström case, however, the stress–energy tensor $T_{\mu\nu}$ is **not zero everywhere**, even though one still speaks of a “vacuum.”

Explanation:

- In the classical Schwarzschild case (without charge), $T_{\mu\nu} = 0$ in the vacuum outside the mass, since no matter or field is present there. Einstein's field equations thus reduce to the vacuum equations.
- In the Reissner–Nordström case, the vacuum surrounding the central charge still contains an electromagnetic field. This field carries energy and momentum, and thus corresponds to a nonzero stress–energy tensor.
- Specifically, $T_{\mu\nu}$ describes the energy–momentum of the radial electric field. This means that Einstein's equation has the electromagnetic field as its source, even if no ordinary matter is present outside the central object.

In summary: In the Reissner–Nordström metric, $T_{\mu\nu} \neq 0$ in the "vacuum" because the electromagnetic field of the charge is physically real and contains energy.

Derivation of the Reissner–Nordström Metric

The following is a step-by-step derivation of the Reissner–Nordström metric from the Einstein–Maxwell equations. This is the standard procedure in general relativity for determining the spacetime of a spherically symmetric, charged mass.

Step 1: Setup — Metric and Source

We seek a static, spherically symmetric solution in spherical coordinates of the form:

$$ds^2 = c^2 d\tau^2 = A(r)c^2 dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\emptyset^2)$$

Where *A(r)* and *B(r)* are unknown functions of *r*.

- The source is the electromagnetic field of a point charge *Q*.
- The stress–energy tensor of the electromagnetic field (in natural units) is:

$$T_{\mu\nu} = \frac{1}{\mu_0}\left(F_{\mu\alpha} F_\nu^\alpha - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right)$$

where:

- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field-strength tensor,
- A_μ is the four-potential,
- $g_{\mu\nu}$ is the metric tensor,
- $\mu_0 = 4\pi \times 10^{-7}\mathrm{H/m}$ is the magnetic permeability of free space.

Additional explanation of A_μ:

$$A_\mu = \left(\frac{\emptyset}{c}, -A)\right) \text{ or } A^\mu = \left(\frac{\emptyset}{c}, A)\right)$$

where

- $\emptyset =$ elektriscpotential (Volts),
- $A = \left(A_x, A_y, A_z\right)$: magnetic vector potential (in Weber per meter),
- *c* : speed of light.

The **elektromagnetic field-strength tensor** $F_{\mu\nu}$ encodes both the electric and magnetic fields **E** and **B**.

In matrixvorm:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

For a purely radial electric field, the only nonzero component is:

$$F_{tr} = -F_{rt} = \frac{Q}{\mathrm{r}^2}$$

Step 2: Einstein–Maxwell-Equations

Einstein's field equations with an electromagnetic source are:

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

The Maxwell equations in vacuum are:

$$\nabla_\mu F^{\mu\nu} = 0$$

and

$$\nabla_{[\alpha} F_{\beta\gamma]} = 0$$

For the static, spherically symmetric case, this gives:

$$F^{tr} = \frac{Q}{r^2} \frac{1}{\sqrt{A(r)B(r)}}$$

Stap 3: Einstein tensor Components

The nonzero components of the Einstein tensor for this metric are:

$$G_t^t = \frac{B'}{rB^2} + \frac{1}{r^2}\left(1 - \frac{1}{B}\right)$$

$$G_r^r = \frac{A'}{rAB} - \frac{1}{r^2}\left(1 - \frac{1}{B}\right)$$

$$G_\theta^\theta = G_\phi^\phi = \frac{1}{4AB}\left(2A'' - \frac{A'B'}{B} + \frac{A'^2}{A}\right) - \frac{1}{2rB}\left(\frac{A'}{A} - \frac{B'}{B}\right)$$

where primes denote derivatives with respect to *r*.

Stap 4: Stress-Energy Tensor Components

For a purely electric field, the tensor is diagonal:

$$T_t^t = T_r^r = -\frac{Q^2}{8\pi r^4}, \ \ T_\theta^\theta = T_\phi^\phi = \frac{Q^2}{8\pi r^4}$$

Stap 5: Solving the Equations

The Einstein equations become:

$$\frac{B'}{rB^2} + \frac{1}{r^2}\left(1 - \frac{1}{B}\right) = -\frac{Q^2}{r^4}$$

$$\frac{A'}{rAB} - \frac{1}{r^2}\left(1 - \frac{1}{B}\right) = -\frac{Q^2}{r^4}$$

Solving yields:

$$A(r) = \frac{1}{B(r)} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$$

where *M* is an integration constant representing the mass (in geometric units).

Stap 6: Final Result — The Reissner–Nordström Metric

$$ds^2 = \left(1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{4\pi\varepsilon_0 c^4 \mathrm{r}^2}\right)c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{4\pi\varepsilon_0 c^4 \mathrm{r}^2}\right)^{-1} dr^2 - r^2 d\Omega^2$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\emptyset^2$

Conclusie

The Reissner–Nordström metric is the unique static, spherically symmetric solution to the Einstein–Maxwell equations for a point mass with electric charge. It incorporates both gravitational and electromagnetic interactions in the spacetime description.

Remark on the Cosmological Constant

The classical Schwarzschild solution is an exact solution of Einstein's field equations under the explicit assumption that the cosmological constant $\lambda = 0$. In Schwarzschild's original derivation, the λ -term is omitted, meaning that the metric does not account for cosmological expansion or repulsion due to a nonzero λ.

How λ included?

- The "standard" Schwarzschild metric is a static vacuum solution without a cosmological constant:

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\emptyset^2$$

$$of$$

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{\mathrm{r_s}}{r}\right)c^2 dt^2 - \left(1 - \frac{\mathrm{r_s}}{r}\right)^{-1} dr^2 - r^2 d\Omega^2$$

where $r_s = \frac{2GM}{c^2}$ and $d\Omega^2 = \ (d\theta^2 + \sin^2\theta\, d\emptyset^2)$.

- The full Einstein equation is:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

but in the Schwarzschild derivation one sets $\lambda = 0$.

If $\lambda \neq 0$: The Schwarzschild–de Sitter (Kottler) metric

When the cosmological constant is included, the **Schwarzschild–de Sitter** (or **Kottler**) metric arises:

$$ds^2 = \left(1 - \frac{\mathrm{r_s}}{r} - \frac{\lambda r^2}{3}\right)c^2 dt^2 - \left(1 - \frac{\mathrm{r_s}}{r} - \frac{\lambda r^2}{3}\right)^{-1} dr^2 - r^2 d\Omega^2$$

This is also an exact solution, now explicitly including λ, and describes, for instance, a black hole in an expanding universe.

Final Summary

- **The Schwarzschild metric is exact, but only for $\lambda = 0$.**
- **For $\lambda \neq 0$, the effect is fully included in the corresponding Schwarzschild–de Sitter solution.**
- Neglecting λ is typically justified for stars or planets, since its value is extremely small compared to local gravitational fields.

Thus, the classical Schwarzschild metric is exact **only under the assumption** that the cosmological constant plays no significant role.

Reissner–Nordström–de Sitter-metric

When we take one step further and incorporate the **Schwarzschild–de Sitter metric** into the **Reissner–Nordström metric**, we obtain the **Reissner–Nordström–de Sitter metric:**

$$ds^2 = \left(1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{4\pi\varepsilon_0 c^4 r^2} - \frac{\lambda r^2}{3}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{4\pi\varepsilon_0 c^4 r^2} - \frac{\lambda r^2}{3}\right)^{-1} dr^2 - r^2 d\Omega^2$$

Explanation of the Terms

- $\frac{2GM}{c^2 r}$: **gravitational (mass) term** — attraction due to mass
- $\frac{GQ^2}{4\pi\varepsilon_0 c^4 r^2}$: **electromagnetic (charge) term** — repulsive for like charges
- $\frac{\lambda r^2}{3}$: **cosmological term** — repulsive for $\lambda > 0$ (de Sitter), attractive for $\lambda < 0$ (anti-de Sitter)

Special cases

- $\boldsymbol{\lambda = 0}$ -> ordinary **Reissner–Nordström-metriek**
- **Q = 0** -> **Schwarzschild–de Sitter** (or Kottler) metric
- **Q = 0, $\boldsymbol{\lambda = 0}$** -> classical **Schwarzschild-metric**

Appendix 5 Schwarzschild Solution Inside a Mass Distribution (ρ = const.) - Complete Tensor Derivation

Appendix 5.1 Introduction

In this appendix we present the full tensor derivation of the internal Schwarzschild solution: the solution of the Einstein field equations for a static, spherically symmetric mass distribution with constant density ρ. We work in Schwarzschild coordinates, with the metric:

$$ds^2 = e^{\nu(r)} c^2 dt^2 - e^{\lambda(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \tag{1}$$

where $\nu(r)$ and $\lambda(r)$ are functions of the radial coordinate r that must be determined.

The energy–momentum tensor of a perfect fluid is:

$$T_{\mu\nu} = (\rho c^2 + p) u_\mu u_\nu - p g_{\mu\nu} \tag{2}$$

Here:

- The first part $(\rho c^2 + p)u_\mu u_\nu$, represents the contribution from moving energy and pressure.
- The second part, $-pg_{\mu\nu}$, ensures that the tensor transforms correctly under Lorentz invariance and reflects the isotropy of a perfect fluid.
- ρ is the mass density (in the rest frame of the fluid).
- ρc^2 is the energy density of the matter.
- p is the isotropic pressure.
- $u^\mu = \frac{dx^\mu}{d\tau}$ is the four-velocity of the matter, with normalization $u^\mu u_\mu = 1$.
- $g_{\mu\nu}$ is the metric tensor.

Where

$$g_{\mu\nu} = \begin{pmatrix} e^\nu & 0 & 0 & 0 \\ 0 & -e^\lambda & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix}$$

Pressure *p* appears in the tensor because in general relativity not only energy but also pressure and stress contribute to the curvature of spacetime. Pressure is a form of energy per unit volume and must therefore be included in the total energy content of the system.

In an ideal (isotropic) rest frame of the fluid, expression (2) reduces to:

$$T_{\mu\nu} = (\rho c^2 + p)u_\mu u_\nu - pg_{\mu\nu} = \begin{pmatrix} \rho c^2 e^\nu & 0 & 0 & 0 \\ 0 & pe^\lambda & 0 & 0 \\ 0 & 0 & pr^2 & 0 \\ 0 & 0 & 0 & pr^2 \sin^2\theta \end{pmatrix}$$

From this it is immediately clear that ρc^2 represents the energy density, and that the spatial diagonal elements correspond exactly to the isotropic pressure *p*. The structure of (1.2) follows directly from the requirements of isotropy and Lorentz invariance: the term $(\rho c^2 + p)u_\mu u_\nu$ represents the energy and momentum density along the direction of motion, while the term $-pg_{\mu\nu}$ represents the isotropic pressure, which is the same in all spatial directions.

For a static fluid we have

$$u^\mu = \left(e^{-\nu/2}, 0,0,0\right), \quad u_\mu = \left(e^{\nu/2}, 0,0,0\right) \quad => \quad u_\mu u_\mu = (e^\nu, 0,0,0)$$

so that:

$$u^\mu u_\mu = 1, \qquad u_\mu u_\nu = \begin{pmatrix} e^\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

In other words, all components vanish except the (*t*,*t*) component;
specifically, $u_t u_t = e^\nu$

This also explains why the time–time component $\rho c^2 e^\nu$ appears in the coordinate expression of $T_{\mu\nu}$:

$$T_{00} = (\rho c^2 + p)u_t u_t - p g_{tt} = \rho c^2 e^{\nu}$$

and

$$T_{rr} = (\rho c^2 + p)u_r u_r - p g_{rr} = -p\left(-e^{\lambda}\right) = p e^{\lambda}$$
$$T_{\theta\theta} = (\rho c^2 + p)u_\theta u_\theta - p g_{\theta\theta} = -p\left(-r^2\right) = p r^2$$
$$T_{\emptyset\emptyset} = (\rho c^2 + p)u_\emptyset u_\emptyset - p g_{\emptyset\emptyset} = -p\left(-r^2 \sin^2\theta\right) = p r^2 \sin^2\theta$$

The Einstein field equations take their usual form:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{4}$$

Explanation

The metric (1) is the most general static, spherically symmetric metric.
The function $\nu(r)$ determines the gravitational redshift (time dilation) in the field, while
$\lambda(r)$ characterizes the curvature in the radial direction.
By determining these two functions from the field equations, we obtain the complete geometric structure of the interior gravitational field.

Appendix 5.2 Computation of the Christoffel-symbols

The Christoffel symbols are defined by:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right\}$$

For the interior Schwarzschild metric:

$$ds^2 = e^{\nu(r)} c^2 dt^2 - e^{\lambda(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\emptyset^2$$

the metric components and their radial derivatives are:

$$g_{\mu\nu} = \begin{pmatrix} e^{\nu} & 0 & 0 & 0 \\ 0 & -e^{\lambda} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix} \qquad \frac{dg_{\mu\nu}}{dx^r} = \begin{pmatrix} \nu' e^{\nu} & -0 & 0 & 0 \\ 0 & -\lambda' e^{\lambda} & 0 & 0 \\ 0 & 0 & -2\mathrm{r} & 0 \\ 0 & 0 & 0 & -2r \sin^2\theta \end{pmatrix}$$

For this metric (with $\nu' = d\nu/dr$, and $\lambda' = d\lambda/dr$), the non-zero Christoffel symbols are:

$$\Gamma^{t}_{tr} = \frac{1}{2} g^{00}\left\{\frac{\partial g_{00}}{\partial x^1}\right\} = \frac{1}{2}\frac{1}{e^{\nu}} e^{\nu}\frac{\partial \nu}{\partial r} = \frac{1}{2}\nu' \tag{5a}$$

$$\Gamma^{r}_{tt} = \frac{1}{2} g^{11}\left\{-\frac{\partial g_{00}}{\partial x^1}\right\} = \frac{1}{2}\left\{\frac{1}{-e^{\lambda}}\right\}\left\{-e^{\nu}\frac{\partial \nu}{\partial r}\right\} = \frac{1}{2} e^{\nu-\lambda}\nu' \tag{5b}$$

$$\Gamma^{r}_{rr} = \frac{1}{2} g^{11}\left\{\frac{\partial g_{11}}{\partial r}\right\} = \frac{1}{2}\left\{\frac{1}{-e^{\lambda}}\right\}\left\{-e^{\lambda}\frac{\partial \lambda}{\partial r}\right\} = \frac{1}{2}\lambda' \tag{5c}$$

$$\Gamma^r_{\theta\theta} = \frac{1}{2} g^{11} \left\{ -\frac{\partial g_{22}}{\partial r} \right\} = \frac{1}{2} \left\{ \frac{1}{-e^{\lambda}} \right\} \{2\mathrm{r}\} = -re^{-\lambda} \qquad (5d)$$

$$\Gamma^r_{\phi\phi} = \frac{1}{2} g^{11} \left\{ -\frac{\partial g_{33}}{\partial r} \right\} = \frac{1}{2} \left\{ \frac{1}{-e^{\lambda}} \right\} \{2r \sin^2\theta\} = -re^{-\lambda} \sin^2\theta \qquad (5e)$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{2} g^{22} \left\{ \frac{\partial g_{22}}{\partial r} \right\} = -\frac{1}{2} \left\{ \frac{1}{-r^2} \right\} 2\mathrm{r} = \frac{1}{\mathrm{r}} \qquad (5f)$$

$$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{2} g^{33} \left\{ \frac{\partial g_{33}}{\partial r} \right\} = -\frac{1}{2} \frac{1}{-r^2 \sin^2\theta} 2r \sin^2\theta = \frac{1}{\mathrm{r}} \qquad (5f)$$

$$\Gamma^\theta_{\phi\phi} = \frac{1}{2} g^{22} \left\{ -\frac{\partial g_{33}}{\partial \theta} \right\} = \frac{1}{2} \frac{1}{-r^2} \{2r^2 . sin(\theta) \cos(\theta)\} = -\sin\theta\cos\theta \qquad (5g)$$

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{1}{2} g^{33} \left\{ \frac{\partial g_{33}}{\partial \theta} \right\} = \frac{1}{2} \frac{1}{-r^2 \sin^2\theta} \{-2r^2 . sin(\theta) \cos(\theta)\} = \frac{\cos(\theta)}{sin(\theta)} = \cot\theta \qquad (5h)$$

Derivation of the Christoffel Symbols

$$\frac{\partial \Gamma^t_{tr}}{\partial r} = \frac{\partial \Gamma^t_{rt}}{\partial r} = \frac{\partial \frac{1}{2} \nu'}{\partial r} = \frac{1}{2} \nu''$$

$$\frac{\partial \Gamma^r_{tt}}{\partial r} = \frac{\partial \frac{1}{2} e^{\nu-\lambda} \nu'}{\partial r} = \frac{1}{2} \left(e^{\nu} e^{-\lambda} \nu'^2 - e^{\nu} e^{-\lambda} \lambda' \nu' + e^{\nu} e^{-\lambda} \nu'' \right) = \frac{1}{2} e^{\nu-\lambda} \left(\nu'^2 - \lambda' \nu' + \nu'' \right)$$

$$\frac{\partial \Gamma^r_{rr}}{\partial r} = \frac{\partial \frac{1}{2} \lambda'}{\partial r} = \frac{1}{2} \lambda''$$

$$\frac{\partial \Gamma^r_{\theta\theta}}{\partial r} = \frac{\partial \left(-re^{-\lambda} \right)}{\partial r} = -e^{-\lambda} + re^{-\lambda} \lambda' = (r\lambda' - 1) e^{-\lambda}$$

$$\frac{\partial \Gamma^r_{\phi\phi}}{\partial r} = \frac{\partial \left(-re^{-\lambda} \sin^2\theta \right)}{\partial r} = -e^{-\lambda} \sin^2\theta + re^{-\lambda} \lambda' \sin^2\theta = (r\lambda' - 1) e^{-\lambda} \sin^2\theta$$

$$\frac{\partial \Gamma^\theta_{r\theta}}{\partial r} = \frac{\partial \Gamma^\theta_{\theta r}}{\partial r} = \frac{\partial \Gamma^\phi_{r\phi}}{\partial r} = \frac{\partial \Gamma^\phi_{\phi r}}{\partial r} = \frac{\partial \frac{1}{\mathrm{r}}}{\partial r} = -\frac{1}{r^2}$$

$$\frac{\partial \Gamma^\theta_{\phi\phi}}{\partial \theta} = \frac{\partial (-\sin\theta\cos\theta)}{\partial \theta} = -\cos^2\theta + \sin^2\theta$$

$$\frac{\partial \Gamma^\phi_{\theta\phi}}{\partial \theta} = \frac{\partial \Gamma^\phi_{\phi\theta}}{\partial \theta} = \frac{\partial \frac{\cos(\theta)}{sin(\theta)}}{\partial \theta} = \frac{-sin^2(\theta)}{sin^2(\theta)} - \frac{\cos^2(\theta)}{\sin^2\theta} = \frac{-1}{\sin^2\theta}$$

Explanation

The derivation of these symbols follows directly from the definition:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \right)$$

and only derivatives with respect to *r* appear, since the metric depends solely on *r*.

Appendix 5.3 Ricci Tensor Components

Ricci Tensor Components

As derived in Chapter **Error! Reference source not found.**, equation **Error! Reference source not found.**, the Ricci tensor defined by:

$$R_{\mu\nu} = R^{\sigma}_{\mu\sigma\nu} = \left(\frac{d\Gamma^{\sigma}_{\mu\nu}}{dx^{\sigma}} - \frac{d\Gamma^{\sigma}_{\mu\sigma}}{dx^{\nu}} + \Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\sigma}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}\right)$$

Full Derivation of R_{tt}

Set $\mu = \nu = t$. We compute the four terms separately.

1. $\frac{\partial \Gamma^{\sigma}_{tt}}{\partial x^{\sigma}}$

Only the $\sigma = r$ term contributes (the other $\Gamma^{\sigma}_{tt} = 0$).

Using

$$\Gamma^{r}_{tt} = \frac{1}{2} e^{\nu-\lambda}\nu'$$

We apply the product rule:

$$\frac{\partial \Gamma^{\sigma}_{tt}}{\partial x^{\sigma}} = \frac{\partial \Gamma^{r}_{tt}}{\partial x^{r}} = \frac{\partial\left(\frac{1}{2}e^{\nu-\lambda}\nu'\right)}{\partial x^{r}} = \frac{1}{2}e^{\nu-\lambda}(\nu'' + (\nu' - \lambda')\nu')$$
$$= \frac{1}{2}e^{\nu-\lambda}(\nu'' + (\nu')^2 - \lambda'\nu')$$

2. $-\frac{\partial \Gamma^{\sigma}_{t\sigma}}{\partial x^{t}}$

Because the metric is static (no time dependence):

$$-\frac{\partial \Gamma^{\sigma}_{t\sigma}}{\partial x^{t}} = 0$$

3. $\Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{tt}$

Only $\lambda = r$ contributes:

$$\Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{tt} = \Gamma^{\sigma}_{\sigma r}\Gamma^{r}_{tt}$$

First compute:

$$\Gamma^{\sigma}_{\sigma r} = \Gamma^{t}_{tr} + \Gamma^{r}_{rr} + \Gamma^{\theta}_{\theta r} + \Gamma^{\phi}_{\phi r} = \frac{1}{2}\nu' + \frac{1}{2}\lambda' + \frac{1}{\mathrm{r}} + \frac{1}{\mathrm{r}} = \frac{1}{2}(\nu' + \lambda') + \frac{2}{\mathrm{r}}$$

Thus:

$$\Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{tt} = \left(\frac{1}{2}(\nu' + \lambda') + \frac{2}{\mathrm{r}}\right)\cdot\frac{1}{2}e^{\nu-\lambda}\nu'$$
$$= \frac{1}{4}e^{\nu-\lambda}\nu'(\nu' + \lambda') + e^{\nu-\lambda}\frac{\nu'}{\mathrm{r}}$$

4. $-\Gamma^{\sigma}_{t\lambda}\Gamma^{\lambda}_{t\sigma}$

Only the pairs involving

$$\Gamma^{\sigma}_{t\lambda} = \frac{1}{2}\nu', \qquad \Gamma^{r}_{tt} = \frac{1}{2}e^{\nu-\lambda}\nu'$$

contribute

$$-\Gamma^{\sigma}_{t\lambda}\Gamma^{\lambda}_{t\sigma} = -2\Gamma^{t}_{tr}\Gamma^{r}_{tt}$$
$$= -2\cdot\frac{1}{2}\nu'\cdot\frac{1}{2}e^{\nu-\lambda}\nu'$$
$$= -\frac{1}{2}e^{\nu-\lambda}(\nu')^2$$

5. **Sum of all terms**

$$R_{tt} = \frac{1}{2}e^{\nu-\lambda}(\nu'' + (\nu')^2 - \lambda'\nu') + \frac{1}{4}e^{\nu-\lambda}\nu'(\nu' + \lambda') + e^{\nu-\lambda}\frac{\nu'}{\mathrm{r}} - \frac{1}{2}e^{\nu-\lambda}(\nu')^2$$

Simplifying gives the standard form:

$$\boldsymbol{R_{tt} = \frac{1}{2}e^{\nu-\lambda}\left(\nu'' + \frac{1}{2}(\nu')^2 - \frac{1}{2}\lambda'\nu' + \frac{2\nu'}{\mathrm{r}}\right)}$$

Full Derivation of R_{rr}

$$R_{rr} = \left(\frac{d\Gamma^{\sigma}_{rr}}{dx^{\sigma}} - \frac{d\Gamma^{\sigma}_{r\sigma}}{dx^{r}} + \Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{rr} - \Gamma^{\sigma}_{r\lambda}\Gamma^{\lambda}_{r\sigma}\right)$$

Compute each term:

1. $\boldsymbol{\frac{\partial\Gamma^{\sigma}_{rr}}{\partial x^{\sigma}}}$

Only $\Gamma^{r}_{rr} = \frac{1}{2}\lambda'$ contributes:

$$\frac{\partial\Gamma^{\sigma}_{rr}}{\partial x^{\sigma}} = \frac{\partial\Gamma^{r}_{rr}}{\partial x^{r}} = \frac{\partial\left(\frac{1}{2}\lambda'\right)}{\partial x^{r}} = \frac{1}{2}\lambda''$$

2. $\boldsymbol{-\frac{\partial\Gamma^{\sigma}_{r\sigma}}{\partial x^{r}}}$

We have:

$$\Gamma^{\sigma}_{r\sigma} = \Gamma^{0}_{r0} + \Gamma^{r}_{rr} + \Gamma^{\theta}_{r\theta} + \Gamma^{\phi}_{r\phi} = \frac{1}{2}\nu' + \frac{1}{2}\lambda' + \frac{1}{\mathrm{r}} + \frac{1}{\mathrm{r}} = \frac{1}{2}(\nu' + \lambda') + \frac{2}{\mathrm{r}}$$

Differentiate:

$$\frac{\partial\Gamma^{\sigma}_{r\sigma}}{\partial x^{r}} = \frac{1}{2}(\nu'' + \lambda'') - \frac{2}{\mathrm{r}^2}$$

Thus:

$$-\frac{\partial\Gamma^{\sigma}_{r\sigma}}{\partial x^{r}} = -\frac{1}{2}(\nu'' + \lambda'') + \frac{2}{\mathrm{r}^2}$$

3. $\boldsymbol{\Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{rr}}$

Only $\lambda = r$ contributes:

$$\boldsymbol{\Gamma^{\lambda}_{rr}} = \frac{1}{2}\lambda'$$

So

$$\Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{rr} = \Gamma^{\sigma}_{\sigma r}\Gamma^{r}_{rr} = \left[\frac{1}{2}(\nu' + \lambda') + \frac{2}{\mathrm{r}}\right]\cdot\frac{1}{2}\lambda' = \frac{1}{4}(\nu' + \lambda')\lambda' + \frac{1}{\mathrm{r}}\lambda'$$

4. $\boldsymbol{-\Gamma^{\sigma}_{r\gamma}\Gamma^{\gamma}_{r\sigma}}$

The non-zero products yield:

$$\Gamma^{\sigma}_{r\lambda}\Gamma^{\lambda}_{r\sigma} = \Gamma^{0}_{r0}\Gamma^{0}_{r0} + \Gamma^{r}_{rr}\Gamma^{r}_{rr} + \Gamma^{\theta}_{r\theta}\Gamma^{\theta}_{r\theta} + \Gamma^{\phi}_{r\phi}\Gamma^{\phi}_{r\phi}$$

$$= \left(\frac{1}{2}\nu'\right)^2 + \left(\frac{1}{2}\lambda'\right)^2 + \left(\frac{1}{\mathrm{r}}\right)^2 + \left(\frac{1}{\mathrm{r}}\right)^2 = \frac{1}{4}(\nu')^2 + \frac{1}{4}(\lambda')^2 + \frac{2}{\mathrm{r}^2}$$

Thus

$$-\Gamma^{\sigma}_{r\lambda}\Gamma^{\lambda}_{r\sigma} = -\frac{1}{4}(\nu')^2 - \frac{1}{4}(\lambda')^2 - \frac{2}{\mathrm{r}^2}$$

5. **Sum of all terms**

We add 1) to 4):

$$R_{rr} = \frac{1}{2}\lambda'' + \left[-\frac{1}{2}(\nu'' + \lambda'') + \frac{2}{\mathrm{r}^2}\right] + \left[\frac{1}{4}(\nu' + \lambda')\lambda' + \frac{1}{\mathrm{r}}\lambda'\right] + \left[-\frac{1}{4}\left(\nu'\right)^2 - \frac{1}{4}\left(\lambda'\right)^2 - \frac{2}{\mathrm{r}^2}\right]$$

The $+2/\mathrm{r}^2$ and $-2/\mathrm{r}^2$concel, as do $\frac{1}{2}\lambda''$ and $-\frac{1}{2}\lambda''$. Which yields to:

$$R_{rr} = -\frac{1}{2}\nu'' + \frac{1}{4}(\nu' + \lambda')\lambda' + \frac{1}{\mathrm{r}}\lambda' + -\frac{1}{4}\left(\nu'\right)^2 - \frac{1}{4}\left(\lambda'\right)^2$$

$$= -\frac{1}{2}\nu'' + \frac{1}{4}\nu'\lambda' + \frac{1}{4}\left(\lambda'\right)^2 + \frac{1}{\mathrm{r}}\lambda' - \frac{1}{4}\left(\nu'\right)^2 - \frac{1}{4}\left(\lambda'\right)^2$$

$$= -\frac{\nu''}{2} - \frac{\left(\nu'\right)^2}{4} + \frac{\nu'\lambda'}{4} + \frac{\lambda'}{\mathrm{r}}$$

Everything simplifies to:

$$\boldsymbol{R_{rr} = -\frac{1}{2}\left(\nu'' + \frac{1}{2}\left(\nu'\right)^2 - \frac{1}{2}\nu'\lambda' - \frac{2}{\mathrm{r}}\lambda'\right)}$$

Full Derivation of $R_{\theta\theta}$

$$R_{\theta\theta} = \left(\frac{d\Gamma^{\sigma}_{\theta\theta}}{dx^{\sigma}} - \frac{d\Gamma^{\sigma}_{\theta\sigma}}{dx^{\theta}} + \Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{\theta\theta} - \Gamma^{\sigma}_{\theta\lambda}\Gamma^{\lambda}_{\theta\sigma}\right)$$

We set $\mu = \nu = \theta$.

1. $\frac{\partial \Gamma^{\sigma}_{\theta\theta}}{\partial x^{\sigma}}$

Only $\sigma = r$ contributes because $\Gamma^{\sigma}_{\theta\theta} = -r\mathrm{e}^{-\lambda}$ (see 5d). So:

$$\frac{\partial \Gamma^{\sigma}_{\theta\theta}}{\partial x^{\sigma}} = \frac{\partial \Gamma^{r}_{\theta\theta}}{\partial x^{r}} = \frac{\partial\left(-r\mathrm{e}^{-\lambda}\right)}{\partial x^{r}} = -\mathrm{e}^{-\lambda} + (-r)\left(-\mathrm{e}^{-\lambda}\right)\lambda' = -\mathrm{e}^{-\lambda} + r\mathrm{e}^{-\lambda}\lambda'$$

2. $-\frac{\partial \Gamma^{\sigma}_{\theta\sigma}}{\partial x^{\theta}}$

Only $\sigma = \phi$ contributes because $\Gamma^{\phi}_{\theta\phi} = \cot\theta$ (see 5h). So:

$$-\frac{\partial \Gamma^{\phi}_{\theta\phi}}{\partial x^{\theta}} = -\frac{\partial \cot\theta}{\partial x^{\theta}} = \frac{1}{\sin^2\theta}$$

3. $\Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{\theta\theta}$

For $\lambda = r$ is $\Gamma^{r}_{\theta\theta} = -\mathrm{r}\mathrm{e}^{-\lambda}$, so:

$$\Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{\theta\theta} = \Gamma^{\sigma}_{\sigma r}\Gamma^{r}_{\theta\theta} = \left(\Gamma^{t}_{tr} + \Gamma^{r}_{rr} + \Gamma^{\theta}_{\theta r} + \Gamma^{\phi}_{\phi r}\right)\Gamma^{r}_{\theta\theta}$$

$$= \left(\frac{1}{2}\nu' + \frac{1}{2}\lambda' + \frac{1}{\mathrm{r}} + \frac{1}{\mathrm{r}}\right)\Gamma^{r}_{\theta\theta} = \left(\frac{1}{2}(\nu' + \lambda') + \frac{2}{\mathrm{r}}\right)\cdot\left(-\mathrm{r}\mathrm{e}^{-\lambda}\right)$$

yields:

$$= -\frac{1}{2}\mathrm{r}\mathrm{e}^{-\lambda}(\nu' + \lambda') - 2\mathrm{e}^{-\lambda}$$

4. $-\Gamma^{\sigma}_{\theta\lambda}\Gamma^{\lambda}_{\theta\sigma}$

$$\begin{aligned}\Gamma^{\sigma}_{\theta\lambda}\Gamma^{\lambda}_{\theta\sigma} = {} & \Gamma^{t}_{\theta t}\Gamma^{t}_{\theta t} + \Gamma^{t}_{\theta r}\Gamma^{r}_{\theta t} + \Gamma^{t}_{\theta\theta}\Gamma^{\theta}_{\theta t} + \Gamma^{t}_{\theta\phi}\Gamma^{\phi}_{\theta t} \\ & + \Gamma^{r}_{\theta t}\Gamma^{t}_{\theta r} + \Gamma^{r}_{\theta r}\Gamma^{r}_{\theta r} + \Gamma^{r}_{\theta\theta}\Gamma^{\theta}_{\theta r} + \Gamma^{r}_{\theta\phi}\Gamma^{\phi}_{\theta r} \\ & + \Gamma^{\theta}_{\theta t}\Gamma^{t}_{\theta\theta} + \Gamma^{\theta}_{\theta r}\Gamma^{r}_{\theta\theta} + \Gamma^{\theta}_{\theta\theta}\Gamma^{\theta}_{\theta\theta} + \Gamma^{\theta}_{\theta\phi}\Gamma^{\phi}_{\theta\theta} \\ & + \Gamma^{\phi}_{\theta t}\Gamma^{t}_{\theta\phi} + \Gamma^{\phi}_{\theta r}\Gamma^{r}_{\theta\phi} + \Gamma^{\phi}_{\theta\theta}\Gamma^{\theta}_{\theta\phi} + \Gamma^{\phi}_{\theta\phi}\Gamma^{\phi}_{\theta\phi}\end{aligned}$$

$$\begin{aligned}\Gamma^{\sigma}_{\theta\lambda}\Gamma^{\lambda}_{\theta\sigma} = {} & 0 + 0 + 0 + 0 \\ & + 0 + 0 + \left(-re^{-\lambda}\right)\left(\frac{1}{\mathrm{r}}\right) + 0 \\ & + 0 + \left(\frac{1}{\mathrm{r}}\right)\left(-re^{-\lambda}\right) + 0 + 0 \\ & + 0 + 0 + 0 + \cot^2\theta\end{aligned}$$

$$\Gamma^{\sigma}_{\theta\lambda}\Gamma^{\lambda}_{\theta\sigma} = 2\left(-re^{-\lambda}\right)\left(\frac{1}{\mathrm{r}}\right) + \cot^2\theta$$

$$-\Gamma^{\sigma}_{\theta\lambda}\Gamma^{\lambda}_{\theta\sigma} = 2\left(re^{-\lambda}\right)\left(\frac{1}{\mathrm{r}}\right) - \cot^2\theta = 2e^{-\lambda} - \cot^2\theta$$

5. Sum of all terms

$$R_{\theta\theta} = \left(-\mathrm{e}^{-\lambda} + r\mathrm{e}^{-\lambda}\lambda'\right) + \left(\frac{1}{\sin^2\theta}\right) + \left(-\frac{1}{2}\mathrm{r}\mathrm{e}^{-\lambda}(\nu' + \lambda') - 2\mathrm{e}^{-\lambda}\right) + \left(2e^{-\lambda} - \cot^2\theta\right)$$

$$1 - \mathrm{e}^{-\lambda} + r\mathrm{e}^{-\lambda}\lambda' - \frac{1}{2}\mathrm{r}\mathrm{e}^{-\lambda}(\nu' + \lambda')$$

$$1 + e^{-\lambda}\left(-1 + r\lambda' - \frac{1}{2}r(\nu' + \lambda')\right)$$

$$1 - e^{-\lambda}\left(1 - r\lambda' + \frac{1}{2}r(\nu' + \lambda')\right)$$

The computation (radial and angular parts) yields the well-known result:

$$\boldsymbol{R_{\theta\theta} = 1 - e^{-\lambda}\left(1 + \frac{r}{2}(\nu' - \lambda')\right)}$$

Full Derivation of $R_{\phi\phi}$

$$R_{\phi\phi} = \left(\frac{d\Gamma^{\sigma}_{\phi\phi}}{dx^{\sigma}} - \frac{d\Gamma^{\sigma}_{\phi\sigma}}{dx^{\phi}} + \Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{\phi\phi} - \Gamma^{\sigma}_{\phi\lambda}\Gamma^{\lambda}_{\phi\sigma}\right)$$

Set $\mu = \nu = \phi$.

1. $\boldsymbol{\frac{\partial\Gamma^{\sigma}_{\phi\phi}}{\partial x^{\sigma}}}$

Only $\sigma = r$ and $\sigma = \theta$ contributes (see 5d and 5g). So:

$$\frac{\partial\Gamma^{\sigma}_{\phi\phi}}{\partial x^{\sigma}} = \frac{\partial\Gamma^{r}_{\phi\phi}}{\partial x^{r}} + \frac{\partial\Gamma^{\theta}_{\phi\phi}}{\partial x^{\theta}} = \frac{\partial\left(-re^{-\lambda}\sin^2\theta\right)}{\partial x^{r}} + \frac{\partial(-\sin\theta\cos\theta)}{\partial x^{\theta}} = -e^{-\lambda}\sin^2\theta\,(1 - r\lambda') - \cos^2\theta + \sin^2\theta$$

2. $\boldsymbol{-\frac{\partial\Gamma^{\sigma}_{\phi\sigma}}{\partial x^{\phi}}}$

No relevant contributions:

$$-\frac{\partial\Gamma^{\sigma}_{\phi\sigma}}{\partial x^{\phi}} = 0$$

3. $\boldsymbol{\Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{\phi\phi}}$

For $\lambda = r$ and $\lambda = \theta$ is $\Gamma^{r}_{\phi\phi} = -re^{-\lambda}$ and $\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$, so:

$$\Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{\phi\phi} = \Gamma^{\sigma}_{\sigma r}\Gamma^{r}_{\theta\theta} + \Gamma^{\sigma}_{\sigma\theta}\Gamma^{\theta}_{\phi\phi} = \left(\Gamma^{t}_{tr} + \Gamma^{r}_{rr} + \Gamma^{\theta}_{\theta r} + \Gamma^{\phi}_{\phi r}\right)\Gamma^{r}_{\phi\phi} + \left(\Gamma^{t}_{t\theta} + \Gamma^{r}_{r\theta} + \Gamma^{\theta}_{\theta\theta} + \Gamma^{\phi}_{\phi\theta}\right)\Gamma^{\theta}_{\phi\phi}$$

$$= \left(\frac{1}{2}\nu' + \frac{1}{2}\lambda' + \frac{1}{r} + \frac{1}{r}\right)\Gamma^{r}_{\phi\phi} + (\cot\theta)\Gamma^{\theta}_{\phi\phi} = \left(\frac{1}{2}(\nu' + \lambda') + \frac{2}{r}\right)\cdot\left(-re^{-\lambda}\sin^2\theta\right) + (\cot\theta)\cdot(-\sin\theta\cos\theta)$$

gives:

$$\Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{\phi\phi} = -e^{-\lambda}\sin^2\theta\left(\frac{r(\nu' + \lambda')}{2} + 2\right) - \cos^2\theta$$

4. $\boldsymbol{-\Gamma^{\sigma}_{\phi\lambda}\Gamma^{\lambda}_{\phi\sigma}}$

$$\Gamma^{\sigma}_{\phi\lambda}\Gamma^{\lambda}_{\phi\sigma} = \Gamma^{t}_{\phi t}\Gamma^{t}_{\phi t} + \Gamma^{t}_{\phi r}\Gamma^{r}_{\phi t} + \Gamma^{t}_{\phi\theta}\Gamma^{\theta}_{\phi t} + \Gamma^{t}_{\phi\phi}\Gamma^{\phi}_{\phi t}$$

$$+\Gamma^r_{\phi t}\Gamma^t_{\phi r} + \Gamma^r_{\phi r}\Gamma^r_{\phi r} + \Gamma^r_{\phi\theta}\Gamma^\theta_{\phi r} + \Gamma^r_{\phi\phi}\Gamma^\phi_{\phi r}$$
$$+\Gamma^\theta_{\phi t}\Gamma^t_{\phi\theta} + \Gamma^\theta_{\phi r}\Gamma^r_{\phi\theta} + \Gamma^\theta_{\phi\theta}\Gamma^\theta_{\phi\theta} + \Gamma^\theta_{\phi\phi}\Gamma^\phi_{\phi\theta}$$
$$+\Gamma^\phi_{\phi t}\Gamma^t_{\phi\phi} + \Gamma^\phi_{\phi r}\Gamma^r_{\phi\phi} + \Gamma^\phi_{\phi\theta}\Gamma^\theta_{\phi\phi} + \Gamma^\phi_{\phi\phi}\Gamma^\phi_{\phi\phi}$$

$$\Gamma^\sigma_{\theta\lambda}\Gamma^\lambda_{\theta\sigma} = 0 + 0 + 0 + 0$$
$$+0 + 0 + 0 + \left(-re^{-\lambda}\sin^2\theta\right)\left(\frac{1}{\mathrm{r}}\right)$$
$$+0 + 0 + 0 + (-\sin\theta\cos\theta)(\cot\theta\ )$$
$$+0 + \left(\frac{1}{\mathrm{r}}\right)\left(-re^{-\lambda}\sin^2\theta\right) + (\cot\theta\ )(-\sin\theta\cos\theta) + 0$$
$$\Gamma^\sigma_{\phi\lambda}\Gamma^\lambda_{\phi\sigma} = 2\left(-re^{-\lambda}\sin^2\theta\right)\left(\frac{1}{\mathrm{r}}\right) + 2(-\sin\theta\cos\theta)(\cot\theta\ )$$
$$-\Gamma^\sigma_{\phi\lambda}\Gamma^\lambda_{\phi\sigma} = 2\left(e^{-\lambda}\sin^2\theta\right) + 2\cos^2\theta = 2\left(e^{-\lambda}\sin^2\theta + \cos^2\theta\right)$$

6. Sum of all terms

$$R_{\phi\phi} = \left(-\mathrm{e}^{-\lambda}\sin^2\theta\,(1 - r\lambda') - \cos^2\theta + \sin^2\theta\ \right) + \left(-e^{-\lambda}\sin^2\theta\left(\frac{\mathrm{r}(\nu' + \lambda')}{2} + 2\right) - \cos^2\theta\right)$$
$$+ 2\left(e^{-\lambda}\sin^2\theta + \cos^2\theta\right)$$
$$R_{\phi\phi} = \left(-\mathrm{e}^{-\lambda}\sin^2\theta\,(1 - r\lambda') + \sin^2\theta\ \right) - \left(e^{-\lambda}\sin^2\theta\left(\frac{\mathrm{r}(\nu' + \lambda')}{2} + 2\right)\right) + 2\left(e^{-\lambda}\sin^2\theta\right)$$
$$R_{\phi\phi} = -\sin^2\theta\,\mathrm{e}^{-\lambda}(1 - \mathrm{r}\lambda') + \sin^2\theta\ - \sin^2\theta\,\mathrm{e}^{-\lambda}\left(\frac{\mathrm{r}(\nu' + \lambda')}{2} + 2\right) + 2\left(e^{-\lambda}\sin^2\theta\right)$$
$$R_{\phi\phi} = \sin^2\theta\,\mathrm{e}^{-\lambda}\left(-1 + \mathrm{r}\lambda' - \frac{\mathrm{r}(\nu' + \lambda')}{2} - 2 + 2\right) + \sin^2\theta$$
$$R_{\phi\phi} = \sin^2\theta\ + \sin^2\theta\,\mathrm{e}^{-\lambda}\left(-1 + \mathrm{r}\lambda' - \frac{\mathrm{r}(\nu' + \lambda')}{2}\right)$$
$$R_{\phi\phi} = \sin^2\theta\ + \frac{\sin^2\theta\,\mathrm{e}^{-\lambda}}{2}(\mathrm{r}\lambda' - \mathrm{r}\nu' - 2)$$
$$\boldsymbol{R_{\phi\phi} = \sin^2\theta\ \left(1 + \frac{\mathrm{e}^{-\lambda}}{2}(\mathrm{r}\lambda' - \mathrm{r}\nu' - 2)\right)}$$

As seen before:

$$R_{\theta\theta} = 1 + \frac{\mathrm{e}^{-\lambda}}{2}(\mathrm{r}\lambda' - \mathrm{r}\nu' - 2)$$

So we see, as required by sperical symmetry, that:

$$R_{\phi\phi} = \sin^2\theta\, R_{\theta\theta}$$

Ricci-Scalar

The Ricci-scalar is:

$$R = g^{\mu\nu} R_{\mu\nu} = g^{tt} R_{tt} + g^{rr} R_{rr} + g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi}$$

From the above we know that:

$$g^{\mu\nu} = \begin{pmatrix} e^{-\nu} & 0 & 0 & 0 \\ 0 & -e^{-\lambda} & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/(r^2 \sin^2\theta) \end{pmatrix}$$

Because $R_{\phi\phi} = \sin^2\theta \, R_{\theta\theta}$, and $g^{\phi\phi} = -\frac{1}{r^2 sin^2\theta}$, one obtains:

$$g^{\phi\phi} R_{\phi\phi} = -\frac{1}{r^2 sin^2\theta} \cdot \sin^2\theta \, R_{\theta\theta} = -\frac{1}{r^2} R_{\theta\theta} = g^{\theta\theta} R_{\theta\theta}$$

and:

$$R_{tt} = \frac{1}{2} e^{\nu-\lambda} \left[\nu'' + \frac{1}{2}(\nu')^2 - \frac{1}{2}\lambda'\nu' + \frac{2\nu'}{\mathrm{r}}\right]$$

$$R_{rr} = -\frac{1}{2}\left(\nu'' + \frac{1}{2}(\nu')^2 - \frac{1}{2}\nu'\lambda' - \frac{2}{\mathrm{r}}\lambda'\right)$$

$$R_{\theta\theta} = 1 - \mathrm{e}^{-\lambda}\left(1 + \frac{\mathrm{r}}{2}(\nu' - \lambda')\right)$$

$$R_{\phi\phi} = \sin^2\theta \, R_{\theta\theta}$$

Thus:

$$R = g^{\mu\nu} R_{\mu\nu} = g^{tt} R_{tt} + g^{rr} R_{rr} + 2g^{\theta\theta} R_{\theta\theta}$$

$$R = \mathrm{e}^{-\nu} R_{tt} - \mathrm{e}^{-\lambda} R_{rr} - \frac{2}{r^2} R_{\theta\theta}$$

$$R = \mathrm{e}^{-\nu}\left[\frac{1}{2} e^{\nu-\lambda}\left(\nu'' + \frac{1}{2}(\nu')^2 - \frac{1}{2}\lambda'\nu' + \frac{2\nu'}{r}\right)\right] - \mathrm{e}^{-\lambda}\left[-\frac{1}{2}\left(\nu'' + \frac{1}{2}(\nu')^2 - \frac{1}{2}\nu'\lambda' - \frac{2}{\mathrm{r}}\lambda'\right)\right] - \frac{2}{r^2}\left[1 - \mathrm{e}^{-\lambda}\left(1 + \frac{\mathrm{r}}{2}(\nu' - \lambda')\right)\right]$$

$$R = \left[\frac{1}{2} e^{-\lambda}\left(\nu'' + \frac{1}{2}(\nu')^2 - \frac{1}{2}\lambda'\nu' + \frac{2\nu'}{r}\right)\right] - \mathrm{e}^{-\lambda}\left[-\frac{1}{2}\left(\nu'' + \frac{1}{2}(\nu')^2 - \frac{1}{2}\nu'\lambda' - \frac{2\lambda'}{\mathrm{r}}\right)\right] - \frac{2}{r^2}\left[1 - \mathrm{e}^{-\lambda}\left(1 + \frac{\mathrm{r}}{2}(\nu' - \lambda')\right)\right]$$

$$R = e^{-\lambda}\left[\frac{1}{2}\left(\nu'' + \frac{1}{2}(\nu')^2 - \frac{1}{2}\lambda'\nu' + \frac{2\nu'}{r}\right) + \frac{1}{2}\left(\nu'' + \frac{1}{2}(\nu')^2 - \frac{1}{2}\nu'\lambda' - \frac{2\lambda'}{\mathrm{r}}\right)\right] - \frac{2}{r^2}\left[1 - \mathrm{e}^{-\lambda}\left(1 + \frac{\mathrm{r}}{2}(\nu' - \lambda')\right)\right]$$

$$R = e^{-\lambda}\left[\frac{1}{2}\left(2\nu'' + (\nu')^2 - \lambda'\nu' + \frac{2\nu'}{r} - \frac{2\lambda'}{\mathrm{r}}\right)\right] - \frac{2}{r^2}\left[1 - \mathrm{e}^{-\lambda}\left(1 + \frac{\mathrm{r}}{2}(\nu' - \lambda')\right)\right]$$

$$R = e^{-\lambda}\left[\left(\nu'' + \frac{1}{2}(\nu')^2 - \frac{1}{2}\lambda'\nu' + \frac{(\nu' - \lambda')}{r}\right)\right] - \frac{2}{r^2}\left[1 - e^{-\lambda}\left(1 + \frac{r}{2}(\nu' - \lambda')\right)\right]$$

$$R = e^{-\lambda}\left[\left(\nu'' + \frac{1}{2}(\nu')^2 - \frac{1}{2}\lambda'\nu' + \frac{(\nu' - \lambda')}{r}\right) + \frac{2}{r^2}\left(1 + \frac{r}{2}(\nu' - \lambda')\right)\right] - \frac{2}{r^2}$$

$$R = e^{-\lambda}\left[\nu'' + \frac{1}{2}(\nu')^2 - \frac{1}{2}\lambda'\nu' + \frac{2(\nu' - \lambda')}{r} + \frac{2}{r^2}\right] - \frac{2}{r^2}$$

$$R = e^{-\lambda}\left(\nu'' + \frac{1}{2}(\nu')^2 - \frac{1}{2}\lambda'\nu' + \frac{2(\nu' - \lambda')}{r}\right) + \frac{2}{r^2}e^{-\lambda} - \frac{2}{r^2}$$

Final result of the Ricci-scalar:

$$\boldsymbol{R = e^{-\lambda}\left(\nu'' + \frac{1}{2}(\nu')^2 - \frac{1}{2}\lambda'\nu' + \frac{2(\nu' - \lambda')}{r}\right) + \frac{2\left(e^{-\lambda} - 1\right)}{r^2}}$$

Summary

$$R_{tt} = \frac{1}{2}e^{\nu-\lambda}\left(\nu'' + \frac{1}{2}(\nu')^2 - \frac{1}{2}\lambda'\nu' + \frac{2\nu'}{r}\right)$$

$$R_{rr} = -\frac{1}{2}\left(\nu'' + \frac{1}{2}(\nu')^2 - \frac{1}{2}\lambda'\nu' - \frac{2\lambda'}{r}\right) \qquad (6)$$

$$R_{\theta\theta} = 1 - e^{-\lambda}\left(1 + \frac{r}{2}(\nu' - \lambda')\right)$$

$$R_{\phi\phi} = \sin^2\theta\, R_{\theta\theta}$$

$$\boldsymbol{R = e^{-\lambda}\left(\nu'' + \frac{1}{2}(\nu')^2 - \frac{1}{2}\lambda'\nu' + \frac{2(\nu' - \lambda')}{r}\right) + \frac{2\left(e^{-\lambda} - 1\right)}{r^2}} \qquad (7)$$

Appendix 5.4 Explicit Einstein Equations

Insurting (6) and (7) into (4) yields the three independent field rquations:

(i) The *tt*-component:

$$G_{tt} = R_{tt} - \frac{1}{2}Rg_{tt} = \frac{8\pi G}{c^4}(\rho c^2 e^{\nu})$$

$$G_{tt} = \frac{1}{2}e^{\nu-\lambda}\left(\nu'' + \frac{1}{2}(\nu')^2 - \frac{1}{2}\lambda'\nu' + \frac{2\nu'}{r}\right) - \frac{1}{2}e^{\nu}\left[e^{-\lambda}\left(\nu'' + \frac{1}{2}(\nu')^2 - \frac{1}{2}\lambda'\nu' + \frac{2(\nu' - \lambda')}{r}\right) + \frac{2\left(e^{-\lambda} - 1\right)}{r^2}\right]$$

$$G_{tt} = \frac{1}{2}e^{\nu-\lambda}\left(\frac{2\lambda'}{r}\right) - \frac{1}{2}e^{\nu}\left[\frac{2\left(e^{-\lambda} - 1\right)}{r^2}\right] = \frac{\lambda'}{r}e^{\nu-\lambda} - \frac{1}{r^2}e^{\nu-\lambda} + \frac{1}{r^2}e^{\nu} = \frac{1}{r^2}e^{\nu-\lambda}(r\lambda' - 1) + \frac{e^{\nu}}{r^2} =$$

$$= \frac{e^{\nu}}{r^2}\left[e^{-\lambda}(r\lambda' - 1) + 1\right] = \frac{8\pi G}{c^4}(\rho c^2 e^{\nu})$$

$$=> \frac{1}{r^2}\left[1 + e^{-\lambda}(r\lambda' - 1)\right] = \frac{8\pi G}{c^2}\rho$$

Thus:

$$\frac{\boldsymbol{1}}{\boldsymbol{r^2}}\frac{\boldsymbol{d}}{\boldsymbol{dr}}\left[\boldsymbol{r}\left(\boldsymbol{1}-\boldsymbol{e^{-\lambda}}\right)\right]=\frac{\boldsymbol{8\pi G}}{\boldsymbol{c^2}}\boldsymbol{\rho} \tag{8}$$

Expanding (8) gives:

$$\frac{1}{r^2}\left[\left(1-e^{-\lambda}\right)+re^{-\lambda}\lambda^{'}\right]=\frac{1}{r^2}\left[1+e^{-\lambda}\left(r\lambda^{'}-1\right)\right]=\frac{8\pi G}{c^2}\rho$$

Thus (8) is identical to the previous expression.

(ii) The *rr*-component:

$$G_{rr}=R_{rr}-\frac{1}{2}Rg_{rr}=\frac{8\pi G}{c^4}pe^{\lambda}$$

$$G_{rr}=\left[-\frac{1}{2}\left(\nu^{''}+\frac{1}{2}(\nu^{'})^2-\frac{1}{2}\lambda^{'}\nu^{'}-\frac{2\lambda^{'}}{r}\right)\right]-\frac{1}{2}\left(-e^{\lambda}\right)\left[e^{-\lambda}\left(\nu^{''}+\frac{1}{2}(\nu^{'})^2-\frac{1}{2}\lambda^{'}\nu^{'}+\frac{2(\nu^{'}-\lambda^{'})}{r}\right)+\frac{2\left(e^{-\lambda}-1\right)}{r^2}\right]$$

$$=\left[-\frac{1}{2}\left(\nu^{''}+\frac{1}{2}(\nu^{'})^2-\frac{1}{2}\lambda^{'}\nu^{'}-\frac{2\lambda^{'}}{r}\right)\right]+\frac{1}{2}\left[\left(\nu^{''}+\frac{1}{2}(\nu^{'})^2-\frac{1}{2}\lambda^{'}\nu^{'}+\frac{2(\nu^{'}-\lambda^{'})}{r}\right)+\frac{2\left(e^{-\lambda}-1\right)}{r^2}e^{\lambda}\right]$$

$$=\left[-\frac{1}{2}\left(\nu^{''}+\frac{1}{2}(\nu^{'})^2-\frac{1}{2}\lambda^{'}\nu^{'}-\frac{2\lambda^{'}}{r}\right)\right]+\frac{1}{2}\left(\nu^{''}+\frac{1}{2}(\nu^{'})^2-\frac{1}{2}\lambda^{'}\nu^{'}+\frac{2(\nu^{'}-\lambda^{'})}{r}\right)+\frac{\left(e^{-\lambda}-1\right)}{r^2}e^{\lambda}$$

$$=\left[\frac{1}{2}\left(\frac{2(\nu^{'}-\lambda^{'})}{r}+\frac{2\lambda^{'}}{r}\right)\right]+\frac{\left(1-e^{\lambda}\right)}{r^2}=\frac{8\pi G}{c^4}pe^{\lambda}$$

$$=\frac{\nu^{'}}{r}+\frac{1}{r^2}\left(1-e^{\lambda}\right)=\frac{8\pi G}{c^4}pe^{\lambda}$$

$$\frac{\nu^{'}}{r}e^{-\lambda}+\frac{1}{r^2}\left(e^{-\lambda}-1\right)=\frac{8\pi G}{c^4}p \tag{9}$$

(ii) The $\boldsymbol{\theta\theta}$ -component:

$$G_{\theta\theta}=R_{\theta\theta}-\frac{1}{2}Rg_{\theta\theta}=\frac{8\pi G}{c^4}pr^2$$

$$G_{\theta\theta}=1-\mathrm{e}^{-\lambda}\left(1+\frac{\mathrm{r}}{2}(\nu^{'}-\lambda')\right)-\frac{1}{2}(-r^2)\left[e^{-\lambda}\left(\nu^{''}+\frac{1}{2}(\nu^{'})^2-\frac{1}{2}\lambda^{'}\nu^{'}+\frac{2(\nu^{'}-\lambda^{'})}{r}\right)+\frac{2\left(e^{-\lambda}-1\right)}{r^2}\right]$$

$$=1-\mathrm{e}^{-\lambda}\left(1+\frac{\mathrm{r}}{2}(\nu^{'}-\lambda')\right)+\frac{1}{2}r^2\left[e^{-\lambda}\left(\nu^{''}+\frac{1}{2}(\nu^{'})^2-\frac{1}{2}\lambda^{'}\nu^{'}+\frac{2(\nu^{'}-\lambda^{'})}{r}\right)+\frac{2\left(e^{-\lambda}-1\right)}{r^2}\right]$$

$$=1-\mathrm{e}^{-\lambda}\left(1+\frac{\mathrm{r}}{2}(\nu^{'}-\lambda')\right)+\left[\frac{1}{2}r^2e^{-\lambda}\left(\nu^{''}+\frac{1}{2}(\nu^{'})^2-\frac{1}{2}\lambda^{'}\nu^{'}+\frac{2(\nu^{'}-\lambda^{'})}{r}\right)+\mathrm{r}e^{-\lambda}(\nu^{'}-\lambda^{'})+\left(e^{-\lambda}-1\right)\right]$$

$$=1-\mathrm{e}^{-\lambda}\left(1+\frac{\mathrm{r}}{2}(\nu^{'}-\lambda')\right)+\left[\frac{1}{2}r^2e^{-\lambda}\left(\nu^{''}+\frac{1}{2}(\nu^{'})^2-\frac{1}{2}\lambda^{'}\nu^{'}\right)-1+e^{-\lambda}\left(1+\mathrm{r}(\nu^{'}-\lambda^{'})\right)\right]$$

$$=\frac{1}{2}r^2e^{-\lambda}\left(\nu^{''}+\frac{1}{2}(\nu^{'})^2-\frac{1}{2}\lambda^{'}\nu^{'}\right)+1+-\mathrm{e}^{-\lambda}-\frac{\mathrm{r}}{2}\mathrm{e}^{-\lambda}(\nu^{'}-\lambda')-1+e^{-\lambda}+\mathrm{r}e^{-\lambda}(\nu^{'}-\lambda^{'})$$

$$=\frac{1}{2}r^2e^{-\lambda}\left(\nu^{''}+\frac{1}{2}(\nu^{'})^2-\frac{1}{2}\lambda^{'}\nu^{'}+\frac{1}{\mathrm{r}}(\nu^{'}-\lambda')\right)=\frac{8\pi G}{c^4}pr^2$$

$$\frac{1}{2}e^{-\lambda}\left(\nu'' + \frac{1}{2}(\nu')^2 - \frac{1}{2}\lambda'\nu' + \frac{1}{r}(\nu' - \lambda')\right) = \frac{8\pi G}{c^4}p \quad (10)$$

Explanation

The three equations are not independent: equation **(10)** follows mathematically from **(8)**, **(9)**, and the conservation law

$$\nabla_\mu T^{\mu r} = 0$$

Therefore, for determining the functions $\lambda(r), \nu(r)$, and *p(r)*, it is sufficient to work with equations **(8)** and **(9)**.

Appendix 5.5 Integration of the First Equation

Integrating equation (8) yields:

$$\frac{1}{r^2}\frac{d}{dr}\left[r\left(1 - e^{-\lambda}\right)\right] = \frac{8\pi G}{c^2}\rho$$

$$\frac{d}{dr}\left[r\left(1 - e^{-\lambda}\right)\right] = \frac{8\pi G}{c^2}\rho r^2$$

$$\int \frac{d}{dr}\left[r\left(1 - e^{-\lambda}\right)\right]dr = \int \frac{8\pi G}{c^2}\rho r^2 dr$$

$$r\left(1 - e^{-\lambda}\right) = \frac{8\pi G}{3c^2}\rho r^3$$

Hence:

$$e^{-\lambda(r)} = 1 - \frac{8\pi G}{3c^2}\rho r^2 \quad (11)$$

With:

$$m(r) = 4\pi \int_0^r \rho r'^2 dr' = \frac{4}{3}\pi\rho r^3$$

This becomes:

$$e^{-\lambda(r)} = 1 - \frac{2Gm(r)}{c^2 r} \quad (12)$$

Explanation

The function *m*(*r*) represents the enclosed mass within radius *r*. For *r* = *R* one has *m*(*R*) = *M*, ensuring a smooth matching to the exterior Schwarzschild solution.

Appendix 5.6 Energy Conservation and the Tolman-Oppenheimer-Volkoff (TOV) Equation

The conservation law $\nabla_\mu T^{\mu r} = 0$ yields:

$$\frac{dp}{dr} = -\frac{1}{2}(\rho c^2 + p)\nu' \quad (13)$$

Use equation (9) to eliminate ν':

$$\frac{\nu'}{r}e^{-\lambda} + \frac{1}{r^2}\left(e^{-\lambda} - 1\right) = \frac{8\pi G}{c^4}p$$

Hence:

$$\nu' = \frac{\frac{8\pi G}{c^4}pr + \frac{1}{r}\left(1 - e^{-\lambda}\right)}{e^{-\lambda}}$$

From (12):

$$e^{-\lambda(r)} = 1 - \frac{2Gm(r)}{c^2 r} \qquad \text{en} \qquad 1 - e^{-\lambda(r)} = \frac{2Gm(r)}{c^2 r}$$

Substituting these into the previous expression gives:

$$\nu' = \frac{\frac{8\pi G}{c^4}pr + \frac{1}{r}\frac{2Gm}{c^2 r}}{1 - \frac{2Gm}{c^2 r}} = \frac{\frac{8\pi G}{c^4}pr + \frac{2Gm}{c^2 r^2}}{\left(1 - \frac{2Gm}{c^2 r}\right)} = \frac{\frac{8\pi G}{c^2}pr^3 + 2Gm}{c^2 r^2\left(1 - \frac{2Gm}{c^2 r}\right)} = \frac{2G\left(m + \frac{4\pi}{c^2}pr^3\right)}{c^2 r^2\left(1 - \frac{2Gm}{c^2 r}\right)}$$

Thus:

$$\nu' = \frac{2G\left(m(r) + 4\pi r^3 p/c^2\right)}{c^2 r^2\left(1 - 2Gm(r)/(c^2 r)\right)} \tag{14}$$

Combining equations (13) and (14) yields the **Tolman–Oppenheimer–Volkoff (TOV) equation**:

$$\frac{dp}{dr} = -\frac{1}{2}(\rho c^2 + p)\nu' = -(\rho c^2 + p)\frac{G\left(m(r) + 4\pi r^3 p/c^2\right)}{c^2 r^2\left(1 - 2Gm(r)/(c^2 r)\right)}$$

$$\frac{dp}{dr} = -\frac{(\rho c^2 + p)G\left(m + \frac{4\pi r^3 p}{c^2}\right)}{c^2 r^2\left(1 - \frac{2Gm}{c^2 r}\right)} \tag{15}$$

Explanation

Equation (15) describes the condition for **mechanical equilibrium**: gravity pulls inward, while the pressure gradient pushes outward.
For constant density ρ, this equation is exactly solvable.

Appendix 5.7 Solution for Constant Density

With

$$m(r) = \frac{4\pi}{3}\rho r^3$$

equation (15):

$$\frac{dp}{dr} = -\frac{(\rho c^2 + p)\left(4\pi G r(\rho + 3p/c^2)\right)}{3c^2\left(1 - \frac{8\pi G\rho r^2}{3c^2}\right)} \tag{16}$$

Define:

$$x(r) \equiv \frac{p(r)}{\rho c^2} \quad => \quad p = \rho c^2 x \;=>\; dp = \rho c^2 dx \tag{16a}$$

Insert this into (16):

Left-hand side:

$$\frac{dp}{dr} = \rho c^2 \frac{dx}{dr}$$

Right-hand side:

$$\rho + \frac{3p}{c^2} = \rho(1+3x) \quad \text{and} \quad \rho c^2 + p = \rho c^2(1+x)$$

Thus:

$$= -\rho c^2(1+x)\frac{4\pi G r\rho(1+3x)}{3c^2(1-\beta r^2)}$$

$$= -\frac{4\pi G\rho}{3}\cdot\frac{r(1+x)(1+3x)\rho c^2}{c^2(1-\beta r^2)}$$

$$= -\frac{4\pi G\rho}{3}\cdot\frac{r(1+x)(1+3x)\rho}{1-\beta r^2}$$

where we have introduced the constant:

$$\beta \equiv \frac{8\pi G}{3c^2}\rho$$

Hence:

$$\frac{\beta}{2} \equiv \frac{4\pi G}{3c^2}\rho$$

Addition of the left-hand side and the right-hand side together yields:

$$\rho c^2 \frac{dx}{dr} = -\frac{4\pi G\rho}{3}\cdot\frac{r(1+x)(1+3x)}{1-\beta r^2}$$

$$\frac{dx}{dr} = -\frac{4\pi G\rho}{3c^2}\cdot\frac{r(1+x)(1+3x)}{1-\beta r^2} = -\frac{\beta}{2}\cdot\frac{r(1+x)(1+3x)}{1-\beta r^2}$$

Separation of the variables gives:

$$\frac{dx}{(1+x)(1+3x)} = -\frac{\beta}{2}\cdot\frac{r}{1-\beta r^2}\,dr$$

Integration of both sides

Left integral – partial integration:

$$\frac{1}{(1+x)(1+3x)} = \frac{-1}{2}\cdot\frac{1}{1+x} + \frac{3}{2}\cdot\frac{1}{1+3x}$$

For the first term:

$$\int \frac{-1}{2}\cdot\frac{dx}{1+x} = -\frac{1}{2}\ln(1+x) + C_1$$

For the second term:

$$\int \frac{3}{2}\cdot\frac{1}{1+3x} = \frac{3}{2}\cdot\frac{1}{3}\ln(1+3x) + C_2 = \frac{1}{2}\ln(1+3x) + C_2$$

Together:

$$\int \frac{dx}{(1+x)(1+3x)} = -\frac{1}{2}\ln(1+x) + \frac{1}{2}\ln(1+3x) + C_3$$

Or:

$$\int \frac{dx}{(1+x)(1+3x)} = \frac{1}{2}\ln\frac{(1+3x)}{(1+x)} + C_3$$

Right integral:

$$-\frac{\beta}{2}\cdot\int \frac{r}{1-\beta r^2}dr$$

Use the substitution $u = 1-\beta r^2 => du = -2\beta r => rdr = -du/(2\beta)$.

Thus:

$$-\frac{\beta}{2}\int \frac{r}{1-\beta r^2}dr = -\frac{\beta}{2}\cdot\left(-\frac{1}{2\beta}\right)\int \frac{1}{u}du = \frac{1}{4}\ln u = \frac{1}{4}\ln(1-\beta r^2)$$

Combining both sides gives:

$$\frac{1}{2}\ln\frac{(1+3x)}{(1+x)} = \frac{1}{4}\ln(1-\beta r^2) + C_4$$

Multiply with 2:

$$\ln\frac{(1+3x)}{(1+x)} = \frac{1}{2}\ln(1-\beta r^2) + C_5$$

Exponentiating:

$$\frac{1+3x}{1+x} = C_6\sqrt{1-\beta r^2}$$

We definine

$$\alpha(r) \equiv \sqrt{1-\beta r^2}$$

Then:

$$\frac{1+3x}{1+x} = C_6\alpha(r)$$

Determining the integration constant

Use the boundery condition $r = R => p(R) = 0 => x(R) = 0$

$$\frac{1+0}{1+0} = C_6\alpha(r) \quad => \quad 1 = C_6 a$$

Where:

$$a \equiv \alpha(R) = \sqrt{1-\beta R^2} = \sqrt{1-\frac{2GM}{c^2R}}$$

So:

$$C_6 = \frac{1}{a}$$

Thus:

$$\frac{1+3x}{1+x} = \frac{\alpha(r)}{a}$$

Hence:

$$x = \frac{\alpha(r) - a}{3a - \alpha(r)} \tag{16b}$$

Now:

$$\alpha(R) = \sqrt{1 - \beta R^2},$$
$$\beta = \frac{8\pi G}{3c^2}\rho$$

As defined before:

$$\alpha(r) = \sqrt{1 - \frac{8\pi G\rho r^2}{3c^2}} = \sqrt{1 - \beta r^2}, \qquad a = \alpha(R) = \sqrt{1 - \frac{2GM}{c^2 R}} \tag{17}$$

Using (16a) and (16b):

$$p(r) = \rho c^2 x = \rho c^2 \frac{\alpha(r) - a}{3a - \alpha(r)} \tag{18}$$

Remarks:

- At $r = R$: $\alpha(R) = a \ \ => p(R) = 0$
- At $r = 0$: $\alpha(0) = 1 \ \ => p(0) = \rho c^2 \frac{1-a}{3a-1}$

Explanation

This solution satisfies $p(R) = 0$ and gives a finite central pressure as long as 3a > 1. The pressure decreases continuously to zero at the surface, which is physically correct.

Determining Explanation $\boldsymbol{\nu(r)}$

From (13):

$$\frac{dv}{dr} = -\frac{2}{\rho c^2 + p}\frac{dp}{dr} \tag{19}$$

Chain rule:

$$\frac{dv}{dr} = \frac{dv}{dp}\frac{dp}{dr} \quad => \quad \frac{dv}{dp} = -\frac{2}{\rho c^2 + p}$$

Integrate with respect to *P*:

$$v = -2\ln(\rho c^2 + p) + \ln C' = \ln C'(\rho c^2 + p)^{-2} = \ln\frac{C'}{(\rho c^2 + p)^2}$$

Exponentiate:

$$e^{\nu/2} = \frac{C''}{\rho c^2 + p} \qquad C'' > 0$$

From (18)

$$p(r) = \rho c^2 \frac{\alpha(r) - a}{3a - \alpha(r)}$$

We get:

$$\rho c^2 + p = \rho c^2 \left(1 + \frac{\alpha - a}{3a - \alpha}\right) = \rho c^2 \frac{2a}{3a - \alpha(r)}$$

Fill in $e^{\nu/2}$:

$$e^{\nu/2} = \frac{C''}{\rho c^2 + p} = \frac{C''}{\rho c^2} \cdot \frac{3a - \alpha(r)}{2a} \tag{19b}$$

Determine C''

At the surface $r = R$ goes $p(R) = 0$ and $\alpha(R) = a$. There the metric component should satisfy:

$$e^{\nu(R)/2} = a = \sqrt{1 - \frac{2GM}{c^2 R}}.$$

Evaluate (19b) at $r = R$:

$$a = \frac{C''}{\rho c^2} \cdot \frac{3a - a}{2a} = \frac{C''}{\rho c^2}$$

Thus:

$$C'' = a\rho c^2$$

Hence:

$$e^{\nu/2} = \frac{a\rho c^2}{\rho c^2} \cdot \frac{3a - \alpha(r)}{2a}$$

$$e^{\nu(r)/2} = \frac{3}{2}a - \frac{1}{2}\alpha(r) \tag{20}$$

Therefore, the metric time-component is:

$$e^{\nu(r)/2} = \left(\frac{3}{2}a - \frac{1}{2}\sqrt{1 - \frac{8\pi G\rho r^2}{3c^2}}\right) \tag{21}$$

Complete Interior Metric

The interior solution for $0 \le r \le R$ becomes:

$$ds^2 = \left(\frac{3}{2}a - \frac{1}{2}\sqrt{1 - \frac{8\pi G\rho r^2}{3c^2}}\right)^2 c^2 dt^2 - \frac{dr^2}{1 - \frac{8\pi G\rho r^2}{3c^2}} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{22}$$

Using:

$$M = \frac{4}{3}\pi\rho R^3 \quad => \quad \rho = \frac{3M}{4\pi R^3}$$

One finds:

$$\frac{8\pi G\rho r^2}{3c^2} = \frac{2GM}{c^2R}\frac{r^2}{R^2}$$

Inserted into (22):

$$\boldsymbol{ds^2 = \left(\frac{3}{2}\sqrt{1-\frac{2GM}{c^2R}} - \frac{1}{2}\sqrt{1-\frac{2GM}{c^2R}\frac{r^2}{R^2}}\right)^2 c^2dt^2 - \frac{dr^2}{1-\frac{2GM}{c^2R}\frac{r^2}{R^2}} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)} \qquad (22a)$$

Explanation

This metric describes the spacetime inside a homogeneous sphere. At *r* = *R* it matches smoothly onto the exterior Schwarzschild solution:

$$e^{-\lambda(R)} = 1 - \frac{2GM}{c^2R}, \qquad e^{\nu(R)} = 1 - \frac{2GM}{c^2R}$$

Appendix 5.8 Central Pressure and the Buchdahl-limit

The central pressure follows from α(0) = 1:

$$p(0) = \rho c^2 \frac{1-a}{3a-1} \qquad (23)$$

AS given earlier in (17)

$$a = \sqrt{1-\frac{2GM}{c^2R}}$$

When

3a−1=0

The central pressure *p*(0) diverges. This yields the **Buchdahl-limit**:

$$\frac{2GM}{c^2R} = \frac{8}{9} \qquad (24)$$

Explanation

This limit marks the maximum compactness allowed for a stable, static configuration of uniform density.
If this bound is exceeded, the star inevitably collapses into a black hole.

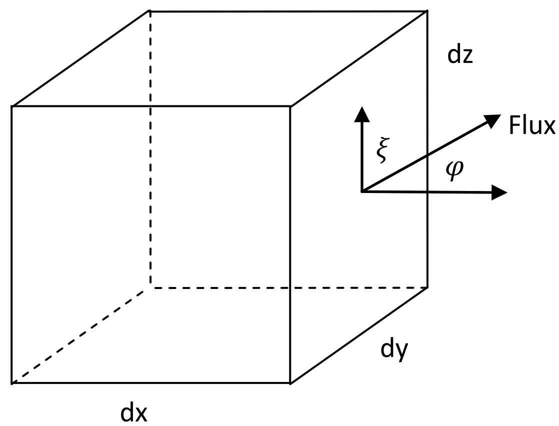
Appendix 5.9 Summary

- The full tensor derivation confirms that the interior Schwarzschild solution is consistent with the Einstein field equations.

- The functions $\nu(r)$ and $\lambda(r)$ can be determined exactly for constant density $\rho = \text{const.}$.
- The pressure $p(r)$ follows from the TOV equation and goes smoothly to $p(R) = 0$.
- The central pressure diverges at the Buchdahl limit $2GM/(c^2R) = 8/9$.
- The interior metric matches smoothly to the exterior Schwarzschild solution at $r=R$.

Appendix 6 Derivation of Gauss's theorem

We start with a cube of infinitesimally small dimensions:



A flux flows through this infinitely small cube F . This flux is not the same everywhere and is therefore a function of *x, y, z*, and *t.* The flux is a vector because it has both a magnitude and a direction.

$$Flux = \vec{F}(x,y,z,t) \tag{1}$$

Flux through a surface

Now consider the right side of the cube, a plane parallel to the y-z plane. The flux flowing through this surface is determined by the component of $\vec{F}$ that is perpendicular to that plane.

If ξ is the angle between $\vec{F}$ and the surface, then:

$$Flux_{rechts} = \vec{F}\sin\xi dydz \tag{2}$$

We represent the surface as a vector $d\vec{A}$, which is perpendicular to the plane:

$$d\vec{A} = \overrightarrow{dy} \times \overrightarrow{dz} \quad \text{with size} \quad dA = \sin\xi dydz \tag{3}$$

The flux flowing through the right side is then:

$$Flux_{rechts} = \vec{F}\sin\xi dydz = \vec{F}\cos\left(\frac{1}{2}\pi - \xi\right)d\vec{A} = \vec{F}\cos\varphi\, d\vec{A} = \vec{F}d\vec{A}\ \cos\varphi \tag{4}$$

The vector $d\vec{A}$ is perpendicular to the surface and φ is the complementary angle of ξ . So here we see the inner product:

$$Flux_{rechts} = \vec{F}d\vec{A}\ \cos\varphi = \vec{F}\cdot d\vec{A} \tag{5}$$

Flux through the total surface of the cube

For a finite cube, the total flux is the sum of the contributions of each surface:

$$Flux_{kubus} = \iint_{right} \vec{F} \cdot d\vec{A} + \iint_{left} \vec{F} \cdot d\vec{A} + \iint_{front} \vec{F} \cdot d\vec{A} + \iint_{back} \vec{F} \cdot d\vec{A} + \iint_{below} \vec{F} \cdot d\vec{A} + \iint_{above} \vec{F} \cdot d\vec{A} \tag{6}$$

We can write this as a single integral over the entire closed surface:

$$Flux_{cube} = \oiint_{cube} \vec{F} \cdot d\vec{A} \tag{7}$$

Alternative approach: flux as a limit

We consider the net flux via a differential approach. In the xxx direction, the flux entering is:

$$Flux_{left} = F_x dydz \tag{8}$$

This flux leaves the right side, increased or decreased by the $d\phi$ from the y or z direction:

$$Flux_{right} = (F_x + dF_x)dydz \tag{9}$$

So, the net flux in the x-direction then becomes:

$$Flux_x = Flux_{right} - Flux_{left} = (F_x + dF_x)dydz - F_x dydz = dF_x dydz \tag{10}$$

The same applies to the y and z directions:

$$Flux_y = dF_y dxdz \tag{11}$$

$$Flux_z = dF_z dxdy \tag{12}$$

The total flux through the cube then becomes:

$$Flux_{cube} = Flux_x + Flux_y + Flux_z = dF_x dydz + dF_y dxdz + dF_z dxdy \tag{13}$$

Or, rewritten as partial derivatives:

$$= \frac{\partial F_x}{\partial x} dxdydz + \frac{\partial F_y}{\partial y} dxdydz + \frac{\partial F_z}{\partial z} dxdydz = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dxdydz$$

$$= \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dV$$

The operator $\vec{\nabla}$ is:

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \tag{14}$$

Equation (13) then becomes:

$$Flux_{cube} = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dV = \left(\vec{\nabla} \cdot \vec{F}\right) \mathrm{dV} \tag{15}$$

By integrating over the entire cube, we find the net flux through the cube:

$$Flux_{cube} = \iiint_{cube} \left(\vec{\nabla} \cdot \vec{F}\right) \mathrm{dV} \tag{16}$$

Gauss's Theorem

Equations (7) and (16) both give the flux through the cube. It follows that:

$$\oiint_{cube} \vec{F} \cdot d\vec{A} = \iiint_{cube} \vec{\nabla} \cdot \vec{F} \,\mathrm{dV} \tag{17}$$

Since the volume was arbitrary (and not necessarily a cube), this applies to any closed volume:

$$\oiint \boldsymbol{\vec{F} \cdot d\vec{A}} = \iiint \boldsymbol{\vec{\nabla} \cdot \vec{F} \,\mathrm{dV}} \tag{18}$$

This equation is known as Gauss's theorem.

Special case: zero flux

In the special case where the net flux through the closed surface is zero (nothing is generated or disappears within the volume):

$$\oiint \vec{F} \cdot d\vec{A} = 0 \quad => \quad \iiint \vec{\nabla} \cdot \vec{F} \,\mathrm{dV} = 0 \tag{19}$$

It follows that:

$$\vec{\nabla} \cdot \vec{F} = 0 \tag{20}$$

This can be written as:

$$\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 0 \tag{21}$$

In **Einstein notation** (with summation over repeated index α):

$$\frac{\partial F^\alpha}{\partial x^\alpha} = 0 \tag{22}$$

Appendix 7 Derivation of the Laplace and Poisson equations

A vector field in which the path between two points does not matter for the work required is called a **conservative field**. In such a field, every route from point A to B costs the same amount of energy. This implies that there exists a scalar potential φ for which the following applies:

$$\vec{F} = \vec{\nabla}\varphi \tag{1}$$

Where the nabla operator $\vec{\nabla}$ is defined as:

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \tag{2}$$

The gravitational field $\vec{F}_g$ is an example of a conservative field, so that:

$$\vec{F}_g = \vec{\nabla}\varphi \tag{3}$$

According to Gauss's theorem, the following applies to every closed surface:

$$\oiint_A \vec{F}_g \cdot d\vec{A} = \iiint_V \vec{\nabla} \cdot \vec{F}_g \, dV \tag{4}$$

In a vacuum, where there is no mass, there is no source of gravity, so that:

$$\vec{\nabla} \cdot \vec{F}_g = 0 \tag{5}$$

Substituting (3) into (5), we obtain:

$$\vec{\nabla} \cdot \vec{F}_g = 0 \quad => \quad \vec{\nabla} \cdot \vec{\nabla}\varphi = 0 \tag{6}$$

or more explicitly:

$$\vec{\nabla} \cdot \vec{\nabla}\varphi = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \varphi = 0 \tag{7}$$

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \varphi = 0$$

Since *x*, *y*, *z* are orthogonal, we are left with:

$$\left(\frac{\partial}{\partial x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\frac{\partial}{\partial z}\right) \varphi = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \varphi = 0$$

This expression is also written as:

$$\boldsymbol{\nabla^2 \varphi = 0} \quad \textbf{of} \quad \boldsymbol{\Delta\varphi = 0} \tag{8}$$

The operator ∇^2 , called the **Laplacian**, is defined as:

$$\boldsymbol{\nabla^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

In vacuum, the **Laplace equation** therefore applies**:**

$$\boldsymbol{\nabla^2\varphi} \textbf{ of } \boldsymbol{\Delta\varphi = 0}.$$

In a volume with mass

Within a mass, there is a source of gravity. According to Newton's law of gravity, the following applies to the gravitational field:

$$\vec{F}_g = G\frac{m}{r^2}\hat{r} \tag{9}$$

where $\hat{r}$ is the unit vector in the radial direction. Apply Gauss's theorem again:

$$\iiint_V \vec{\nabla} \cdot \vec{F}_g dV = \oiint_A \vec{F}_g \cdot d\vec{A} \tag{10}$$

$$\iiint_V \Delta\varphi dV = \oiint_A G\frac{m}{r^2}\hat{r} \cdot d\vec{A} = \oiint_A G\frac{m}{r^2} dA$$

The surface area and volume of a sphere are respectively:

$$A = 4\pi r^2 \tag{11}$$

$$V = \frac{4}{3}\pi r^3 \tag{12}$$

Because the radius *r* of the sphere remains constant over the total surface of the sphere, equation (10) becomes:

$$\iiint_V \Delta\varphi dV = \oiint_A G\frac{m}{r^2}dA = G\frac{m}{r^2}\oiint_A dA = G\frac{m}{r^2}4\pi r^2 = 4\pi Gm \tag{13}$$

With ρ as the mass density:

$$\rho = \frac{m}{V} \tag{14}$$

Then (13) becomes:

$$\iiint_V \Delta\varphi dV = 4\pi Gm = 4\pi G\iiint_V \rho dV = \iiint_V 4\pi G\rho dV \;\; ==> \;\; \Delta\varphi = 4\pi G\rho \tag{15}$$

This is **Poisson's equation**, which applies in areas where mass is present.

Summary

- In a region with mass density ρ :

$$\Delta\varphi = 4\pi G\rho \tag{16}$$

Or:

$$\nabla^2\varphi = 4\pi G\rho \quad (\text{Poisson} - \text{vergelijking})$$

- In empty space (vacuum):

$$\Delta\varphi = 0 \tag{17}$$

Or:

$$\nabla^2\varphi = 0 \qquad (\text{Laplace} - \text{vergelijking})$$

Consideration

The existence of mass causes gravitational flux. When you are inside a mass sphere and move outward, the amount of enclosed mass changes and so does the total flux ($\nabla^2\varphi = 4\pi G\rho$) . When you are finally outside the mass sphere, the mass remains enclosed and the total flux remains constant ($\nabla^2\varphi = 0$) .

Appendix 7.1 Application of the Laplace operator to the gravitational potential

In this chapter, we apply the Laplace operator to the gravitational potential, both outside and inside a static sphere. The relevant formulas for the Newtonian potential are derived in Appendix 7.1.1(outside a sphere) and Appendix 7.1.2(inside a sphere).

According to Newton, gravity is:

$$F = mg = \frac{GmM}{r^2} \Longrightarrow \text{gravitation field: } g = \frac{GM}{r^2} \Rightarrow \text{ gravitation potential: } \emptyset_{newton} = \frac{-\mathrm{G}M}{r}$$

$$\text{where } g = \frac{d\emptyset_{newton}}{dr}$$

Here, *r* is the distance from the center of the sphere and *R* is the radius of the sphere. *M* is the mass of the sphere and *m* is the mass of a particle.

The gravitational potential **outside** a sphere in general relativity is (Chapter 2.8 equation 5):

$$\emptyset = g_{00} = 1 - \frac{2GM}{c^2 r} = 1 + \frac{2\emptyset_{newton}}{c^2}$$

$$=> \emptyset_{newton\ \_outside} = -\frac{GM}{r} \tag{1}$$

Gravitational potential **inside** a sphere (see derivation below):

$$\emptyset = 1 - \frac{3GM}{c^2 R} + \frac{GM}{c^2}\frac{r^2}{R^3} = 1 + \frac{2}{c^2}\cdot\left(-\frac{3GM}{2R} + \frac{GM}{2}\frac{r^2}{R^3}\right)$$

$$=> \emptyset_{newton\ \_inside} = -\frac{3GM}{2R} + \frac{GM}{2}\frac{r^2}{R^3} \tag{2}$$

See Appendix 7.1.4equation 3.

Next, we apply the Laplace operator to the gravitational potential outside and inside a sphere, where:

$$r^2 = x^2 + y^2 + z^2$$

Appendix 7.1.1 Outside a Sphere (Laplace)

$$r^2 = x^2 + y^2 + z^2$$

The derivative with respect to *x* gives:

$$\frac{\partial r}{\partial x} \Longrightarrow \ 2r\frac{\partial r}{\partial x} = 2\mathrm{x} \Longrightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

The gravitational potential outside a sphere is given in classical mechanics by (see equation 1 in Appendix 7.1):

$$\emptyset_{newton\ \_outside} = -\frac{GM}{r}$$

First derivative with respect to *x*:

$$\frac{\partial \emptyset_{newton}}{\partial x} = \frac{\partial \emptyset_{newton}}{\partial r}\cdot\frac{\partial r}{\partial x} = \frac{GM}{r^2}\cdot\frac{x}{r} = \frac{GMx}{r^3}$$

Second derivative with respect to *x:*

$$\frac{\partial^2 \emptyset_{newton}}{\partial x^2} = \frac{-3GMx}{r^4}\cdot\frac{x}{r} + \frac{GM}{r^3} = \frac{-3GMx^2}{r^5} + \frac{GM}{r^3}$$

The same applies to *y* and *z*. So, in total for *x*, *y*, and *z,* the following applies:

$$\Delta\emptyset_{newton} = \frac{\partial^2 \emptyset_{newton}}{\partial x^2} + \frac{\partial^2 \emptyset_{newton}}{\partial y^2} + \frac{\partial^2 \emptyset_{newton}}{\partial z^2}$$

$$\Delta\emptyset_{newton} = \frac{-3GM}{r^3}\cdot\frac{x^2+y^2+z^2}{r^2} + \frac{3GM}{r^3} = \frac{-3GM}{r^3} + \frac{3GM}{r^3} = 0$$

Therefore:

$$\Delta\emptyset_{newton} = 0$$

Outside a sphere without mass (vacuum), the gravitational potential therefore satisfies the Laplace equation.

Appendix 7.1.2 Inside a sphere (Poisson)

As derived above:

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

The gravitational potential inside a homogeneous spherical mass is (see equation 2 in Appendix 7.1):

$$\emptyset_{newton\ \_inside} = -\frac{3GM}{2R} + \frac{GM}{2}\frac{r^2}{R^3}$$

$$\frac{\partial \emptyset_{newton}}{\partial x} = \frac{\partial \emptyset_{newton}}{\partial r}\frac{\partial r}{\partial x} = \frac{2GM}{2}\frac{r}{R^3}\frac{x}{r} = \frac{GMx}{R^3}$$

$$\frac{\partial^2 \emptyset_{newton}}{\partial x^2} = \frac{GM}{R^3}$$

The same applies to *y* and *z.* So in total*:*

$$\Delta\emptyset_{newton} = \frac{\partial^2 \emptyset_{newton}}{\partial x^2} + \frac{\partial^2 \emptyset_{newton}}{\partial y^2} + \frac{\partial^2 \emptyset_{newton}}{\partial z^2} = \frac{3GM}{R^3} = \frac{3G.\frac{4}{3}\pi R^3\rho}{R^3} = 4\pi G\rho$$

Within the sphere, the gravitational potential therefore satisfies the **Poisson equation**:

$$\Delta\emptyset_{\boldsymbol{newton}} = \boldsymbol{4\pi G\rho} \qquad (\mathbf{3})$$

Therefore:

$$\emptyset = 1 + \frac{2\emptyset_{newton}}{c^2} \Longrightarrow \Delta\emptyset = \frac{2}{c^2}\Delta\emptyset_{newton} = \frac{2}{c^2}4\pi G\rho = \frac{8\pi G\rho}{c^2}$$

$$\Delta\emptyset = \boldsymbol{\frac{8\pi G\rho}{c^2}}$$

Appendix 7.1.3 Simplification of the Application of the Laplace/Poisson Operator

Let us assume that we have a function *f(r)* to which the Laplace operator is applied.

$$r^2 = x^2 + y^2 + z^2$$

Gradient of $f(r)$:

$$\nabla f(r) = \left(\frac{\partial f(r)}{\partial x}, \frac{\partial f(r)}{\partial y}, \frac{\partial f(r)}{\partial z}\right)$$

$$\frac{\partial f(r)}{\partial x} = \frac{\partial f(r)}{\partial r}\cdot\frac{\partial r}{\partial x} = \frac{\partial f(r)}{\partial r}\cdot\frac{\vec{x}}{r} \qquad (1)$$

The gradient of $f(r)$:

$$\nabla f(r) = \frac{\partial f(r)}{\partial r}\cdot\left(\frac{\vec{x}}{r} + \frac{\vec{y}}{r} + \frac{\vec{z}}{r}\right) = \frac{\partial f(r)}{\partial r}\cdot\frac{\vec{r}}{r} = \frac{\partial f(r)}{\partial r}\cdot\hat{r}$$

Further differentiation of (1):

$$\frac{\partial^2 f(r)}{\partial x^2} = \frac{\partial^2 f(r)}{\partial r^2} \cdot \frac{x}{r} \cdot \frac{x}{r} + \frac{\partial f(r)}{\partial r} \cdot \frac{1}{r} - \frac{\partial f(r)}{\partial r} \cdot \frac{x}{r^2} \cdot \frac{x}{r}$$

$$\frac{\partial^2 f(r)}{\partial x^2} = \frac{\partial^2 f(r)}{\partial r^2} \cdot \frac{x^2}{r^2} + \frac{\partial f(r)}{\partial r} \cdot \frac{1}{r} \cdot (1 - \frac{x^2}{r^2})$$

Now for x, y, and z:

$$\frac{\partial^2 f(r)}{\partial x^2} + \frac{\partial^2 f(r)}{\partial y^2} + \frac{\partial^2 f(r)}{\partial z^2} = \frac{\partial^2 f(r)}{\partial r^2} \cdot \frac{x^2 + y^2 + z^2}{r^2} + \frac{\partial f(r)}{\partial r} \cdot \frac{1}{r} \cdot (3 - \frac{x^2 + y^2 + z^2}{r^2})$$

The Laplace/Poisson equation:

$$\Delta \boldsymbol{f}(\boldsymbol{r}) = \frac{\boldsymbol{\partial^2 f(r)}}{\boldsymbol{\partial r^2}} + \frac{\mathbf{2}}{\boldsymbol{r}} \cdot \frac{\boldsymbol{\partial f(r)}}{\boldsymbol{\partial r}} \qquad (\mathbf{2})$$

Let us take the general form of $\emptyset_{newton}$:

$$\emptyset_{newton} = L + Kr^n \qquad (3)$$

Where L and K are constants.

$$\frac{\partial \emptyset_{newton}}{\partial r} = nKr^{n-1}$$

$$\frac{\partial^2 \emptyset_{newton}}{\partial r^2} = n(n-1)Kr^{n-2}$$

Therefore, according to equation (2):

$$\Delta \emptyset_{newton} = n(n-1)Kr^{n-2} + \frac{2}{r} \cdot nKr^{n-1} = n(n-1)Kr^{n-2} + 2nKr^{n-2}$$

$$\Delta \emptyset_{\boldsymbol{newton}} = \boldsymbol{n}(\boldsymbol{n}+\mathbf{1})\boldsymbol{K}\boldsymbol{r}^{\boldsymbol{n}-\mathbf{2}} \qquad (\mathbf{4})$$

Let's apply this formula to the gravitational potentials outside and inside a sphere.

Outside a sphere:

$$\emptyset_{\boldsymbol{newton}} = -\frac{\boldsymbol{GM}}{\boldsymbol{r}}$$

So, according to (3)

$$\emptyset_{newton} = L + Kr^n$$

Where n = -1, L = 0, and K = -GM. Then, according to (4):

$$\Delta \emptyset_{newton} = -1(-1+1)GMr^{-1-2} = 0 \cdot GMr^{-3} = 0$$

Inside a sphere:

$$\emptyset_{newton} = -\frac{\mathbf{3GM}}{\mathbf{2R}} + \frac{\mathbf{GM}}{\mathbf{2}}\frac{\mathbf{r^2}}{\mathbf{R^3}}$$

So, according to (3)

$$\emptyset_{newton} = L + Kr^n$$

Where n=+2, L=-3GM/2R, and K=GM/2R^3

$$\Delta\emptyset_{newton} = +2(2+1)\frac{GM}{2R^3}r^{2-2} = 6\frac{GM}{2R^3} = \frac{\mathbf{3GM}}{\mathbf{R^3}} = \frac{3G.\frac{4}{3}\pi R^3\rho}{R^3} = \mathbf{4\pi G\rho}$$

This corresponds to the calculations in the previous chapter.
Furthermore, it can be seen that $\Delta\emptyset_{newton}$ is zero when n=0 or -1, and of course when r goes to infinity while n < 2.

Appendix 7.1.4 Derivation of the Gravitational Potential Within a Static Sphere

The gravitational potential within a static sphere will be derived based on Poisson's equation:

$$\Delta\emptyset_{\mathbf{newton}} = \mathbf{4\pi G\rho}.$$

And the general form of $\emptyset_{newton}$:

$$\emptyset_{newton} = L + Kr^n$$

With formula (4) derived above:

$$\Delta\emptyset_{\mathbf{newton}} = \boldsymbol{n(n+1)Kr^{n-2}} \qquad (2)$$

It follows that:

$$4\pi G\rho = n(n+1)Kr^{n-2}$$

This indicates that when n=2, the following applies:

$$6K = 4\pi G\rho => K = \frac{2}{3}\pi G\rho = \frac{2}{3}\pi G\frac{M}{\frac{4}{3}\pi R^3} = \frac{1}{2}\frac{GM}{R^3}$$

So the gravitational potential inside a static sphere is:

$$=> \emptyset_{newton} = L + \frac{2}{3}\pi G\rho r^2$$

On the surface of the sphere, where *r=R:*

$$\emptyset_{newton} = -\frac{GM}{R}$$

For a continuous transition from Ø on the surface of the sphere (at *r=R*), the outer gravitational potential must be equal to the inner gravitational potential:

$$\emptyset_{newton} = -\frac{GM}{R} = -\frac{4}{3}\pi\frac{R^3}{R}G\rho = -\frac{4}{3}\pi R^2 G\rho = L + \frac{2}{3}\pi G\rho R^2$$

This gives:

$$L = -\frac{4}{3}\pi R^2 G\rho - \frac{2}{3}\pi G\rho R^2 = -\frac{6}{3}\pi R^2 G\rho = -\frac{6}{3}\pi R^2 G\frac{M}{\frac{4}{3}\pi R^3} = -\frac{3}{2}\frac{MG}{R}$$

The gravitational potential inside the sphere then becomes:

$$\emptyset_{newton} = L + \frac{2}{3}\pi G\rho r^2 = L + \frac{2}{3}\pi G r^2 \frac{M}{\frac{4}{3}\pi R^3} = L + \frac{1}{2}\frac{GM}{R^3}r^2$$

So:

$$\emptyset_{newton} = -\frac{3}{2}\frac{MG}{R} + \frac{1}{2}\frac{GM}{R^3}r^2$$

The acceleration g_r is the derivative of $\emptyset_{newton}$ with respect to *r*:

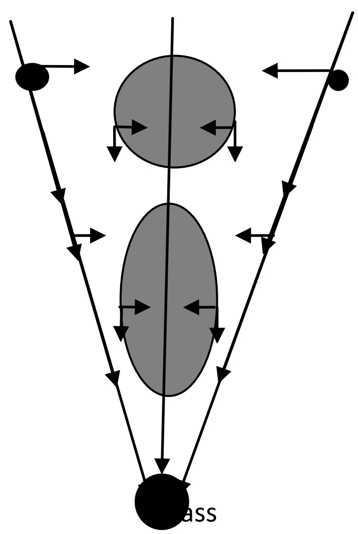
$$g_r = \frac{d\emptyset_{newton}}{dr} = \frac{GM}{R^3}r$$

At *r*=0, the acceleration is $g_r = 0$ and at *r=R,* the acceleration is $g_r = \frac{GM}{R^2}$.

Gravitational potential inside the sphere:

$$\emptyset = 1 + \frac{2\emptyset_{newton}}{c^2} = \boldsymbol{1 - \frac{3MG}{c^2R} + \frac{GM}{c^2R^3}r^2} \qquad (3)$$

Appendix 8 Tidal forces



Tidal forces

The lines of the gravitational field caused by a mass are not parallel but directed toward the center of the mass. The magnitude of the force is inversely proportional to the square of the distance from the center of the mass. The gravitational forces on the gray body can be split into horizontal and vertical components. The gray body is compressed by the horizontal components of the force, and the body is stretched vertically because the gravitational field increases as we get closer to the mass.

So, since the lines of the gravitational field are radially oriented, the force is called a tidal force.

In the case of a "black hole," the forces are so enormous that the gray body is stretched to such an extent that this phenomenon is called "spaghettification."

Appendix 9 Special Theory of Relativity

In the Special Theory of Relativity, Einstein only considered coordinate systems that moved uniformly, i.e., at a constant speed relative to each other; the influence of masses, and therefore gravity, was not taken into account. The assumptions on which the Special Theory of Relativity is based are:

- The maximum possible speed, in any coordinate system, is the speed of light *c*=299 792 458 m/s.
- The laws of nature are valid in every uniformly moving coordinate system.

In Newton's approach, the time intervals were the same in the "rest frame" and in the moving frame. However, Special Relativity showed that the **time intervals** in a moving frame are different and smaller than in a rest frame. In addition, the **length of an object** is influenced by its speed and decreases, relative to the rest frame, in the direction of motion.

Both were consequences of the observation that the speed of light in a vacuum is always the same in every system, regardless of the speed of the system.

In this chapter, we summarize a number of points that are often used in Special Relativity (SR) and that are relevant for application in general relativity.
We begin by establishing the relationship between two coordinate systems moving at a constant speed relative to each other. This relationship is known as the Lorentz transformation, the derivation of which is shown below.

Appendix 9.1 Simple Derivation of the Lorentz Transformation

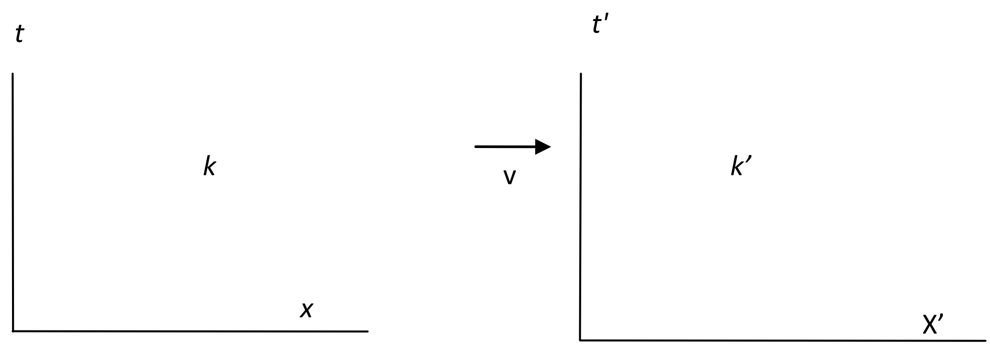


Fig. 1
Coordinate system k' moves uniformly at a speed v relative to coordinate system k.

We take two coordinate systems whose origins move relative to each other at a constant speed *v*, in the *x* and *x'* directions, respectively. Although the coordinate systems are four-dimensional (*t, x, y, z*), only the t and x axes are drawn for simplicity, because there is no movement in the y and z directions.

A light signal is emitted at time $t = t^{'} = 0$ in the direction of the positive x-axis, according to the equation:

$$x = ct$$

Or:

$$x - ct = 0 \tag{1}$$

Since the same light signal is emitted at speed *c* relative to *k'*, propagation relative to the *k'* system will be represented by the analogous formula:

$$x' - ct' = 0 \tag{2}$$

Those space-time points (events) that satisfy (1) must also satisfy (2). This is clearly the case when the relationship:

$$(x' - ct') = \lambda(x - ct) \tag{3}$$

generally applies, where λ denotes a constant; because according to (3), the disappearance of ($x - ct$) leads to the disappearance of ($x' - ct'$) for every value of lambda.

If we apply similar considerations to light rays emitted along the negative x-axis, we obtain the condition:

$$(x' + ct') = \mu(x + ct) \tag{4}$$

By adding (or subtracting) equations (3) and (4) and inserting the constants *a* and *b* instead of λ and μ , where:

$$a = \frac{\lambda + \mu}{2}$$

and

$$b = \frac{\lambda - \mu}{2}$$

we obtain the equations:

$$\begin{aligned} x' &= ax - bct \\ ct' &= act - bx \end{aligned} \tag{5}$$

We would therefore have the solution to our problem if the constants *a* and *b* were known. These follow from the following discussion.

For the origin of *k'*, $x' = 0$ applies permanently, and therefore, according to the first of the equations (5), we have

$$x = \frac{bc}{a}t$$

If we call *v* the speed at which the origin of *k'* moves relative to *k,* then we have:

$$v = \frac{bc}{a} \tag{6}$$

The same value *v* can be obtained from equation (5) if we calculate the velocity of another point of *k'* relative to *k*, or the velocity (directed toward the negative x-axis) of a point of *k* relative to *k'.* In short, we can refer to *v* as the relative velocity of the two systems.

Furthermore, the principle of relativity teaches us that, as judged from *k,* the length of a ruler that is at rest relative to *k'* must be exactly the same as the length, as judged from *k',* of a ruler that is at rest relative to *k.* To see how the points on the x'-axis look from k, we only need to take a "snapshot" of *k'* from k*;* this means that we must enter a certain value of *t* (the time of k*)*, for example t= 0. For this value of t, we then obtain from the first of the equations (5):

$$x' = ax$$

Two points on the *x*'-axis, separated by the distance $x' = L$ when measured in the *k*'-system, are therefore separated in our snapshot by the distance:

$$\Delta x = \frac{L}{a} \tag{7}$$

But if the snapshot is taken from *k'*(*t'* = 0), and if we eliminate *t* from equations (5), taking into account expression (6), we obtain:

$$0 = act - bx$$

$$t = \frac{b}{ac}x$$

$$x^{'} = ax - bct = ax - \frac{b^2}{a}x = ax\left(1 - \frac{b^2}{a^2}\right)$$

From (6) we get:

$$\frac{b}{a} = \frac{v}{c}$$

$$=> x^{'} = a\left(1 - \frac{v^2}{c^2}\right)x \tag{7a}$$

From this we conclude that two points on the x-axis, separated by the distance *L* (relative to *k*), are represented in our snapshot by the distance:

$$\Delta x^{'} = a\left(1 - \frac{v^2}{c^2}\right)L \tag{7b}$$

But from what has been said, the two snapshots must be identical; so Δx in (7) must be equal to $\Delta x^{'}$ in (*7b*), so that we obtain:

$$\Delta x = \frac{L}{a} = \Delta x^{'} = a\left(1 - \frac{v^2}{c^2}\right)L$$

$$\frac{1}{a} = a\left(1 - \frac{v^2}{c^2}\right)$$

$$=> a^2 = \frac{1}{1 - \frac{v^2}{c^2}} \tag{7c}$$

Equations (6) and (*7c*) determine the constants *a* and *b.* By inserting the values of these constants into (5), we obtain the equations:

$$x^{'} = ax - bct = ax - avt = a(x - vt)$$

$$x^{'} = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$ct^{'} = act - bx = act - \frac{av}{c}x = ac\left(t - \frac{v}{c^2}x\right)$$

$$t^{'} = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{8}$$

This gives us the Lorentz transformation for events on the x-axis.
This satisfies the condition:

$$x'^2 - c^2t'^2 = x^2 - c^2t^2 \qquad (8a)$$

The extension of this result to include events outside the x-axis is obtained by retaining equations (8) and supplementing them with the relations:

$$y' = y$$
$$z' = z \qquad (9)$$

In this way, we satisfy the postulate of the constant speed of light *in a vacuum* for light rays in arbitrary directions, both for the system *k* and for the system *k'.* This can be demonstrated as follows.

We assume that a light signal is emitted from the origin of *k* at time *t* = 0. It will propagate according to the equation:

$$r = \sqrt{x^2 + y^2 + z^2} = ct$$

Or, if we square this equation, according to the equation:

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \qquad (10)$$

According to the law of propagation of light, in combination with the postulate of relativity, the transmission of the signal in question - judged from *K*' - must take place according to the corresponding formula:

$$r' = ct'$$

Or,

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0 \qquad (10a)$$

To ensure that equation (10a) is a consequence of equation (10), we must have:

$$(x'^2 + y'^2 + z'^2 - c^2t'^2) = \sigma(x^2 + y^2 + z^2 - c^2t^2) \qquad (11)$$

Since equation (8a) must apply to points on the x-s, we therefore have $\sigma = 1$; because (11) is a consequence of (8a) and (9), and therefore also of (8) and (9). This is how we derived the Lorentz transformation.

The Lorentz transformation, represented by (8) and (9), still needs to be generalized. It is clear that it does not matter whether the axes of *k'* are chosen to be spatially parallel to those of *k.* Nor is it essential that the velocity of the translation of *k'* relative to *k* lies in the direction of the x-axis. A simple consideration shows that we are able to construct the Lorentz transformation in this general sense from two types of transformations, namely from Lorentz transformations in the specific sense and from purely spatial transformations, which corresponds to replacing the rectangular coordinate system with a new system with its axes in different directions.

Mathematically, we can characterize the generalized Lorentz transformation as follows: it expresses *x', y', z', t'* in terms of linear homogeneous functions of *x, y, z, t,* of such a nature that the relation:

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2 \qquad (11a)$$

is satisfied. That is to say: if we substitute their expressions in *x, y, z, t,* instead of *x', y', z', t',* on the left-hand side, the left-hand side of (11a) corresponds to the right-hand side.

We can characterize the Lorentz transformation even more simply if we use the imaginary quantity

$$\sqrt{-1}ct$$

introduce instead of *ct,* as a time variable. If we introduce this:

$$
\begin{aligned}
x_1 &= x \\
x_2 &= y \\
x_3 &= z \\
x_4 &= \sqrt{-1}.ct
\end{aligned}
$$

and do the same for the accentuated system *k'*, then the condition that is identically satisfied by the transformation can be expressed as follows:

$$x_1{'}^2 + x_2{'}^2 + x_3{'}^2 + x_4{'}^2 = x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2 \tag{12}$$

With this choice of "coordinates," (11a) is converted into this equation.

We see from (12) that the imaginary time coordinate x_4 appears in the transformation condition in exactly the same way as the spatial coordinates x_1 , x_2 , x_3 . This is because, according to the theory of relativity, "time" x_4 appears in the laws of nature in the same way as the spatial coordinates x_1 , x_2 , x_3 .

A four-dimensional continuum described by the "coordinates" x_1 , x_2 , x_3 , x_4 , was called "world" by Minkowski, and he called a point event a "world point." From an "event" in three-dimensional space, physics becomes, as it were, an "existence" in the four-dimensional "world."

This four-dimensional "world" bears a strong resemblance to the three-dimensional "space" of (Euclidean) analytical geometry. If we introduce a new Cartesian coordinate system (x'_1, x'_2, x'_3) with the same origin point, then x'_1 , x'_2 , x'_3 , linear homogeneous functions of x_1 , x_2 , x_3 , which identically satisfy the equation:

$$x_1{'}^2 + x_2{'}^2 + x_3{'}^2 = x_1{}^2 + x_2{}^2 + x_3{}^2$$

The analogy with (12) is complete. We can formally consider Minkowski's "world" as a four-dimensional Euclidean space (with imaginary time coordinate); the Lorentz transformation corresponds to a "rotation" of the coordinate system in the four-dimensional "world."

Appendix 9.2 Alternative derivation of time dilation and length contraction

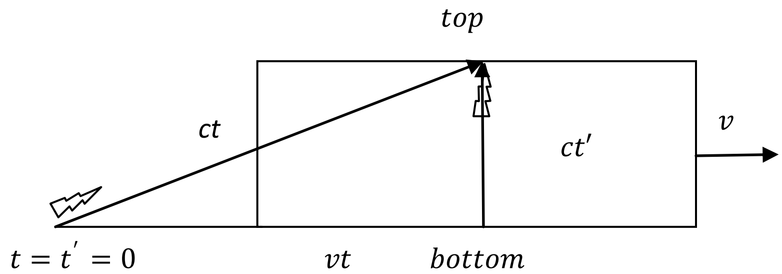
To illustrate the effects of special relativity on time and length, we use a light signal in a rapidly moving object, for example a rocket. We consider two reference frames:

- our stationary reference frame with time *t*,
- the reference frame of the rocket with time *t'*.

Time dilation

First we send a light flash **perpendicular to the direction of motion of the rocket**. In the rocket frame, the light flash moves from the bottom to the top of the rocket. The height of the rocket in that frame is hhh, and the light flash covers this distance in the time t':

$$h = ct'$$



From our stationary reference frame, the rocket appears to move to the right with velocity vvv. The light flash still follows the vertical distance *h*, but because the rocket moves horizontally, the light flash follows a diagonal path in our frame. The horizontal displacement of the rocket is vtvtvt, while the vertical displacement of the light is still *h*. The total distance traveled by the light in our frame is ctctct. From the Pythagorean relation it follows that:

$$c^2t^2 = c^2t'^2 + v^2t^2$$
$$c^2t^2 - v^2t^2 = c^2t'^2$$
$$t^2(c^2 - v^2) = c^2t'^2$$
$$t^2\left(1 - \frac{v^2}{c^2}\right) = t'^2$$
$$t' = t\sqrt{1 - \frac{v^2}{c^2}}$$

From this it follows that a clock in the moving frame runs more slowly: the time t't't' in the rocket is shorter than the time ttt in our stationary frame. This is the phenomenon of **time dilation**.

Length contraction

When measuring a length in a reference frame, the positions of the two ends must be determined **at the same moment in that frame**. In our stationary frame, the positions of the back and the front of the rocket are therefore recorded simultaneously at *t*. In the rocket frame itself, the positions of the two ends are recorded at the same moment *t'*.

Starting from the Lorentz transformations:

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Since *x'* and *t'* are functions of *x* and *t*, it also follows that:

$$dx' = \frac{dx - vdt}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad dt' = \frac{dt - \frac{v}{c^2}dx}{\sqrt{1 - \frac{v^2}{c^2}}}$$

To measure the length in our frame we must keep *t* constant, so *dt*=0, which gives:

$$dx' = \frac{dx}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The length in the rocket is L_0, because it is at rest in that frame, and we then see the length in our frame as:

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}}$$

Because of the relativity of simultaneity, events that are simultaneous in one frame are not necessarily simultaneous in another frame. This explains why a moving rocket, from our perspective, has a shorter length L than the rest length L_0 measured in the rocket frame.

Summary of the results

$$t' = t\sqrt{1 - \frac{v^2}{c^2}}, \; L = L_0\sqrt{1 - \frac{v^2}{c^2}}$$

- A clock in a moving frame runs more slowly (**time dilation**).
- A moving object is shorter in the direction of motion (**length contraction**).
- Length contraction arises from the combination of motion and the relativity of simultaneity.

Appendix 9.3 Symmetry of Lorentz transformations on spacetime

When a Lorentz transformation is performed due to a constant velocity *v* in the *x*-direction, formally only the coordinates *x* and *t* are directly affected::

$$\begin{gathered} x' = \gamma(x - vt) \\ t' = \gamma\left(t - \frac{v}{c^2}x\right) \\ y' = y \\ z' = z \end{gathered}$$

At first glance it appears that only the x−*t* plane changes, while the other spatial coordinates *y* and *z* remain unchanged. However, this is only partially correct.

When we consider spacetime as a four-dimensional structure, we see that **all processes in which the time component *t* appears are indirectly affected**. For example:

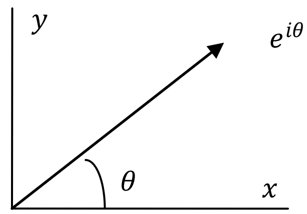
- A clock located at a fixed *y*-position ticks more slowly for an observer at rest, just like a clock at a fixed *x*-position.
- Similarly, events in the *y*−*t* or *z*−*t* plane are affected because the time *t* is transformed.

Conversely, in planes such as *y*−*x* or *z*−*x*, the coordinate *x* is directly transformed. Here too spacetime is altered, although the effect is less directly visible.

The key insight is that the Lorentz transformation is **not limited to the plane of motion**. Every four-dimensional combination in which *x* or *t* appears is affected. This produces a form of structural symmetry: although *y* and *z* themselves do not change, all processes in those directions proceed according to a transformed time or space component. The entire spacetime is restructured, and with it all physical descriptions within that frame.

Appendix 9.4 Trigonometric Tools

Because trigonometric formulas are often used in special relativity, we provide a brief overview of a number of them and how they can be easily derived.



By definition:

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1}$$

Where:

$$i = \sqrt{-1}$$

Substantiation of this equation:

First, we consider a function:

$$F(x) = e^{\alpha x}$$

The derivative is:

$$\frac{d\, e^{\alpha x}}{dx} = \alpha e^{\alpha x}$$

So:

$$\frac{dF(x)}{dx} = \alpha F(x)$$

Therefore, the derivative of a function $F(x) = e^{\alpha x}$ is a factorα times that function.

Next, we consider a function:

$$F(x) = \cos\alpha x + i\sin\alpha x$$

Whose derivative is:

$$\frac{d(\cos\alpha x + i\sin\alpha x)}{dx} = -\alpha\sin\alpha x + i\alpha\cos\alpha x = i\alpha(\cos\alpha x + i\sin\alpha x)$$

Here we see again that:

$$\frac{dF(x)}{dx} = i\alpha F(x)$$

Where:

$$F(x) = e^{i\alpha x} = \cos\alpha x + i\sin\alpha x\,.$$

From this we can deduce:

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad (1)$$

From this equation, all trigonometric formulas can be derived, such as:

$$e^{-i\theta} = \cos\theta - i\sin\theta \qquad (2)$$

By adding (1) and (2), we get:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

And by subtracting (1) and (2), we get:

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Furthermore:

$$e^{i\theta} \cdot e^{-i\theta} = e^{i\theta - i\theta} = e^0 = 1 = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \sin^2\theta + \cos^2\theta = 1$$

Next, we define the hyperbolic functions:

$$\cosh \mathrm{x} = \frac{e^x + e^{-x}}{2}$$
$$\sinh x = \frac{e^x - e^{-x}}{2}$$

From these equations, we can derive:

$$\cosh(\mathrm{x}) = \cosh(-\mathrm{x})$$
$$\sinh(x) = -\sinh(-x)$$
$$\cosh(\mathrm{ix}) = \cos(\mathrm{x})$$
$$-\mathrm{i}\sinh(ix) = \sin(x)$$

With these tools, we should be able to derive all the necessary trigonometric equations.

Appendix 9.5 Addition of velocities

We consider two coordinate systems A and B that move relative to each other at a constant velocity *v* m/s. The coordinate systems are chosen such that the relative motion between the systems occurs along their x-axes. In A, an object moves at velocity *V*' with components in all directions. Now we must consider the velocity of the object relative to system B. According to Newton, the added velocity relative to system B is V_x'+ *v*. However, according to the special theory of relativity, it is different:

First, we start with the equations for the Lorentz transformation, derived in the previous chapters:

$$ct' = \frac{ct - \frac{v}{c}x}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma(ct - \beta x) \qquad (1)$$

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma(x - \beta ct) \qquad (2)$$

$$y' = y$$
$$z' = z$$

Here,

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \text{ and } \beta = \frac{v}{c}$$

The velocity *v* of the origin of system A relative to system B is here in the *x*-direction.
The relationship between system B and A:

$$ct = \gamma(ct' + \beta x') \qquad (1a)$$
$$x = \gamma(x' + \beta ct') \qquad (2a)$$
$$y = y'$$
$$z = z'$$

The velocity in the x' direction in system A can be found by taking the derivative of (2):

$$V_x' = \frac{dx'}{dt'} = \gamma\left(\frac{\partial x}{\partial t}\frac{dt}{dt'} - \beta c\frac{dt}{dt'}\right) = \gamma\left(\frac{\partial x}{\partial t} - \beta c\right)\frac{dt}{dt'} = \gamma(V_x - \beta c)\frac{dt}{dt'} \qquad (3)$$

The derivative of (1):

$$c\frac{dt'}{dt'} = c = \gamma\left(c\frac{dt}{dt'} - \beta\frac{\partial x}{\partial t}\frac{dt}{dt'}\right) = \gamma\left(c - \beta\frac{\partial x}{\partial t}\right)\frac{dt}{dt'} = \gamma(c - \beta V_x)\frac{dt}{dt'}$$

From this it follows:

$$\frac{dt}{dt'} = \frac{1}{\gamma\left(1 - \frac{\beta V_x}{c}\right)} \qquad (4)$$

Substitute (4) into (3):

$$\boldsymbol{V_x'} = \frac{\gamma(V_x - \beta c)}{\gamma\left(1 - \frac{\beta V_x}{c}\right)} = \frac{\boldsymbol{V_x - \beta c}}{\boldsymbol{1 - \frac{\beta V_x}{c}}} \qquad (5)$$

Speed in the *y*'-direction:

$$V_y' = \frac{\partial y'}{\partial t'} = \frac{\partial y}{\partial t'} = \frac{\partial y}{\partial t}\frac{dt}{dt'} = V_y\frac{dt}{dt'} = \frac{V_y}{\gamma\left(1 - \frac{\beta V_x}{c}\right)}$$

So:

$$\boldsymbol{V_y'} = \frac{\boldsymbol{V_y}}{\boldsymbol{\gamma\left(1 - \frac{\beta V_x}{c}\right)}} \qquad (6)$$

Similarly for the z'-direction:

$$\boldsymbol{V_z'} = \frac{\boldsymbol{V_z}}{\boldsymbol{\gamma\left(1 - \frac{\beta V_x}{c}\right)}} \qquad (\mathbf{7})$$

Now look at equation (4):

$$\frac{dt}{dt'} = \frac{1}{\gamma\left(1 - \frac{\beta V_x}{c}\right)} = \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{vV_x}{c^2}}$$

In the special case where $V_x^{'} = 0$ and $V_x = v$:

$$\frac{dt}{dt'} = \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v^2}{c^2}} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow dt^{'} = \sqrt{1 - \frac{v^2}{c^2}}\, dt \text{ so } dt^{'} \ll dt$$

Back to the general case:
For the *X* component (5):

$$V_x^{'} = \frac{V_x - \beta c}{1 - \frac{\beta V_x}{c}} = \frac{V_x - v}{1 - \frac{vV_x}{c^2}}$$

Or:

$$V_x = \frac{V_x^{'} + v}{1 + \frac{vV_x^{'}}{c^2}}$$

A similar derivation via equations (1a) and (2a) gives:

$$\boldsymbol{V_x = \frac{V_x^{'} + \beta c}{1 + \frac{\beta V_x^{'}}{c}}} \qquad \boldsymbol{(5a)}$$

$$\boldsymbol{V_y = \frac{V_y^{'}}{\gamma\left(1 + \frac{\beta V_x^{'}}{c}\right)}} \qquad \boldsymbol{(6a)}$$

$$\boldsymbol{V_z = \frac{V_z^{'}}{\gamma\left(1 + \frac{\beta V_x^{'}}{c}\right)}} \qquad \boldsymbol{(7a)}$$

So, according to Newton, we would have an added velocity in the x-direction of:

$$V_x^{'} + v$$

but according to special relativity, Newton's result is corrected to:

$$\frac{V_x^{'} + v}{1 + \frac{vV_x^{'}}{c^2}}$$

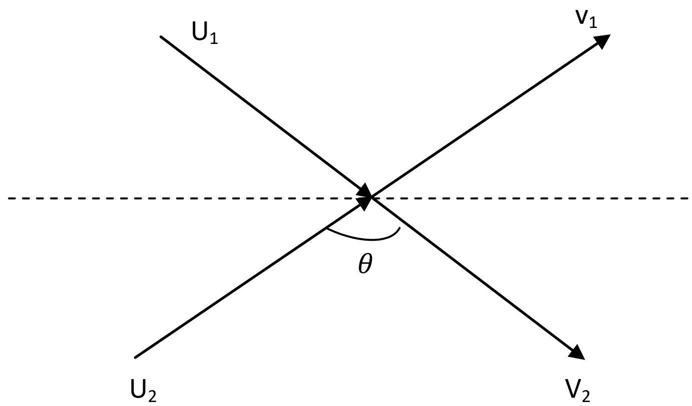
In general, when the term $vV_x^{'}$ is much smaller than c^2, we can approximate the result with Newton's result $V_x^{'} + v$.

Appendix 9.6 Collisions

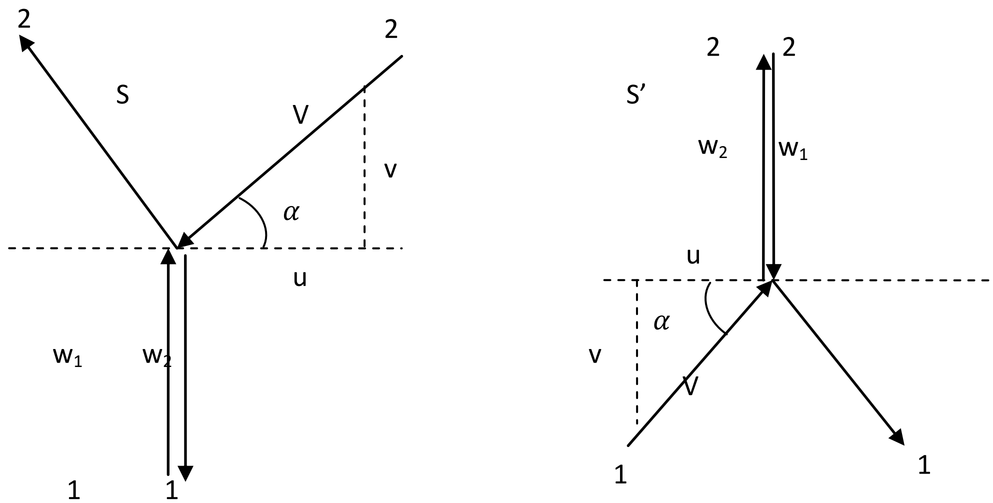
Imagine a perfectly elastic collision between two identical particles; an elastic collision is a collision without loss of kinetic energy. The initial velocities of the particles are $\vec{u}_1$and $\vec{u}_2$ and, after the collision, $\vec{v}_1$ and $\vec{v}_2$. Due to the conservation of momentum, the following applies:

$$m_{1u}u_1 + m_{2u}u_2 = m_{1v}v_1 + m_{2v}v_2$$

Here, m_{1u} and m_{2u} are the masses before the collision and m_{1v} and m_{2v} are the masses after the collision.



First, we consider the collision from a coordinate system that moves with particle one. Then particle 1 moves upward at velocity w_1 and downward at w_2 . These velocities are equal but opposite. Particle 2 has velocity $\vec{V}$ with an x-component *u* and a y-component *v.*



Left: Collision between two identical particles in a coordinate system S that moves with particle 1. Right: The same, but now S' moves with particle 2.

Now we need to find the relationship between the *y*-components of the momentum of particles 1 and 2 in system S, i.e., *w* and *v*. In the previous chapter, we found the following relationship:

$$V_y' = \frac{V_y}{\gamma\left(1 - \frac{\beta V_x}{c}\right)}$$

Since:

$$V_y = w \text{ and } V_x = 0$$

we get:

$$v = \frac{w}{\gamma}$$

Due to symmetry, *w* here is the velocity of particle 1 in system S and the velocity of particle 2 in S'. *v* is the *y*-component of particle 2 in S and of particle 1 in S'.

The total velocity of the moving particle 1 in S and of the moving particle in S' is the same, namely:

$$V = \sqrt{v^2 + u^2}$$

The conservation of momentum in the y-direction now gives:

$$m_w w - m_V v = -m_w w + m_V v$$

from which it follows that:

$$m_w w = m_V v$$

So:

$$\frac{m_V}{m_w} = \frac{w}{v} = \frac{w}{w/\gamma} = \gamma \qquad (1)$$

Now suppose that the velocity w is very small. In this limit, the following applies:

$$lim_{w\to 0} v = 0 \text{ and } lim_{w\to 0} V = u.$$

In that case, the relativistic effects can be neglected and the classical expression for momentum can be derived.

So:

$$lim_{w\to 0} m_w = m$$

Substitute this into (1):

$$lim_{w\to 0} m_V = \gamma m = \frac{m}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Due to momentum conservation, the definition of momentum must be adjusted. This relativistic extension is:

$$\vec{p} = \gamma m \vec{v}$$

Appendix 9.7 The Energy of a Moving Object

With the thought experiment, Einstein demonstrated that energy and mass are equivalent via the relationship $E = mc^2$. We have shown that for an object moving at a velocity, the momentum must be adjusted to the relativistic description:

$$\vec{p} = \gamma m \vec{v}$$

So it can be stated that the energy of an object is equal to:

$$E = \gamma mc^2.$$

Therefore:

$$E = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}}$$

With the Taylor series expansion:

$$E = \gamma mc^2 \approx mc^2 \left(1 + \frac{v^2}{2c^2} - \frac{3v^4}{8c^4} \dots \dots.\right)$$

If v is much smaller than c, the third and subsequent terms within the brackets can be neglected. This leads to:

$$E \approx mc^2 + \frac{1}{2}mv^2$$

So this is the kinetic energy $\frac{1}{2}mv^2$ plus a constant mc^2.

Appendix 9.8 Energy-Momentum Vector

As found by Minkowski:

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \tag{1}$$

$$c^2 d\tau^2 = c^2 dt^2 \left(1 - \frac{dx^2 + dy^2 + dz^2}{c^2 dt^2}\right)$$

$$d\tau^2 = dt^2 \left(1 - \frac{v^2}{c^2}\right)$$

Since:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \;=>\; 1 - \frac{v^2}{c^2} = \frac{1}{\gamma^2}$$

$$d\tau^2 = dt^2 \left(1 - \frac{v^2}{c^2}\right) = \frac{dt^2}{\gamma^2}$$

$$=> \boldsymbol{\gamma = \frac{dt}{d\tau}}$$

From (1) we derive:

$$c^2 = c^2 \frac{dt^2}{d\tau^2} - \frac{dx^2}{dt^2}\frac{dt^2}{d\tau^2} - \frac{dy^2}{dt^2}\frac{dt^2}{d\tau^2} - \frac{dz^2}{dt^2}\frac{dt^2}{d\tau^2}$$

$$m_0{}^2 c^2 = m_0{}^2 c^2 \frac{dt^2}{d\tau^2} - m_0{}^2 \frac{dx^2}{dt^2}\frac{dt^2}{d\tau^2} - m_0{}^2 \frac{dy^2}{dt^2}\frac{dt^2}{d\tau^2} - m_0{}^2 \frac{dz^2}{dt^2}\frac{dt^2}{d\tau^2}$$

$$m_0{}^2 c^2 = \gamma^2 m_0{}^2 c^2 - \gamma^2 m_0{}^2 v_x{}^2 - \gamma^2 m_0{}^2 v_y{}^2 - \gamma^2 m_0{}^2 v_z{}^2$$

$$p^2 = \left(\frac{E}{c}\right)^2 - p_x{}^2 - p_y{}^2 - p_z{}^2$$

$$p_0 = \frac{E}{c}$$

$$p_1 = p_x$$

$$p_2 = p_y$$

$$p_3 = p_z$$

$$p^2 = \left(\frac{E}{c}\right)^2 - |\vec{p}|^2 = m_0{}^2 c^2$$

$$E^2 - c^2 |\vec{p}|^2 = m_0{}^2 c^4$$

$$E = \pm\sqrt{m_0{}^2 c^4 + c^2 |\vec{p}|^2}$$

Or:

$$E^2 = {m_0}^2 c^4 + c^2 |\vec{p}|^2$$

Where:

$$p = \gamma m_0 v = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

And m_0 is the rest mass (mass at zero velocity)

Or via the relationship:

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \quad => \quad E^2 = \frac{{m_0}^2 c^4}{1 - \frac{v^2}{c^2}}$$

Now let's continue:

$$E^2 = \frac{{m_0}^2 c^4}{1 - \frac{v^2}{c^2}} = {m_0}^2 c^4 + \frac{{m_0}^2 c^4 \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}$$

$$=> {m_0}^2 c^4 + \frac{{m_0}^2 v^2 c^2}{1 - \frac{v^2}{c^2}}$$

So:

$$E^2 = {m_0}^2 c^4 + p^2 c^2$$

Or, as is commonly written:

$$\boldsymbol{E^2 = p^2 c^2 + {m_0}^2 c^4} \tag{2}$$

Where:

$$p = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Appendix 9.8.1 Alternative derivation of the Energy-Impulse-Mass relationship

$$p = mv$$

$$p = \gamma m_0 v = \gamma m_0 c^2 \frac{v}{c^2}$$

$$pc = \gamma m_0 c^2 \frac{v}{c} = \beta \gamma m_0 c^2$$

Here is:

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} \ \ and \ \ \beta = \frac{v}{c}$$

Now, using the above, we look at what happens:

$$(pc)^2 + (m_0c^2)^2 = (\beta\gamma m_0c^2)^2 + (m_0c^2)^2$$
$$(pc)^2 + (m_0c^2)^2 = (m_0c^2)^2(\beta^2\gamma^2 + 1)$$
$$(pc)^2 + (m_0c^2)^2 = (m_0c^2)^2\left(1 + \beta^2\frac{1}{1-\beta^2}\right)$$
$$(pc)^2 + (m_0c^2)^2 = (m_0c^2)^2\left(\frac{1-\beta^2+\beta^2}{1-\beta^2}\right) = (m_0c^2)^2\left(\frac{1}{1-\beta^2}\right) = (m_0c^2)^2\gamma^2$$
$$(pc)^2 + (m_0c^2)^2 = (m_0c^2)^2\gamma^2 = (\gamma m_0c^2)^2 = E^2$$

So:

$$\boldsymbol{E^2 = (pc)^2 + \left(m_0c^2\right)^2}$$

Appendix 9.8.2 Classical Proof of Energy Conservation

Energy is the sum of kinetic energy *K* and potential energy *U*:

$$E = \frac{1}{2}mv^2 + U$$

Time derivative and partial derivative for a one-dimensional situation:

$$\frac{dE}{dt} = mv\frac{dv}{dt} + \frac{dU}{dx}\frac{dx}{dt} = mva + v\frac{dU}{dx}$$

The force on a particle, according to the principle of potential energy, is related to the derivative of the potential energy, *U(x)*:

$$F = -\frac{dU}{dx}$$
$$\frac{dE}{dt} = v\left(ma + \frac{dU}{dx}\right) = v(ma - F)$$

We know that according to Newton:

$$F = ma$$

Therefore:

$$\frac{dE}{dt} = 0 => E = constant$$

The energy *E* is therefore conserved.

Appendix 9.9 Derivation of $\mathrm{E = mc^2}$

Einstein found the equation $E=mc^2$ through his so-called thought experiments:

There is a stationary box floating in space, unaffected by any gravitational force. When a photon is emitted on the left side and moves to the right, the box will move slightly to the left due to the law of conservation of momentum. At a certain point, the photon collides with the right side of the box, transferring all its momentum to the box. Due to the law of conservation of momentum, the box stops moving.

The photon has moved and the box has also moved, while no external forces were present. So the center of mass of the system does not change location.

As we saw in the previous chapter, the following applies to relativistic energy (see **Appendix 9.8** equation (2)):

$$E^2 = p^2c^2 + {m_0}^2c^4$$

For a photon, the mass is zero, so:

$$E = pc$$

So the momentum of the photon is:

$$p_{photon} = \frac{E}{c}$$

The box with mass M will move slightly in the opposite direction at velocity *v.*
The momentum of the box is:

$$p_{box} = Mv$$

In the time Δt , the photon will reach the other side. In this time, the box will have moved a distance of Δx . The velocity of the box is:

$$v = -\frac{\Delta x}{\Delta t}$$

Due to the law of conservation of momentum, the following applies $p_{photon} + p_{box} = 0 \;=>\; p_{box} = -p_{photon}$.
So:

$$M\frac{\Delta x}{\Delta t} = \frac{E}{c}$$

The length of the box is *L* and the time it takes for the photon to reach the other side of the box is:

$$\Delta t = \frac{L}{c}$$

So:

$$M\Delta x = \frac{EL}{c^2}$$

Suppose, hypothetically, that the photon has some mass *m*. Then the center of mass of the entire system can be calculated. If the position of the box is x_1 and the photon has position x_2 , then the center of mass of the system is:

$$\bar{x} = \frac{Mx_1 + mx_2}{M + m}$$

It is required that the center of mass of the entire system does not change. So the center of mass must be the same at the end of the experiment as it was at the beginning:

$$\frac{Mx_1 + mx_2}{M + m} = \frac{M(x_1 - \Delta x) + mL}{M + m}$$

The photon starts at $x_2 = 0$, so we get:

$$mL = M\Delta x$$

Now we get:

$$mL = \frac{EL}{c^2}$$

With some rearrangement:

$$\boldsymbol{E = mc^2}$$

Note:

It seems that an approximation has been made in this derivation, because when the photon reaches the other side of the box, the box has moved a little bitΔx in the opposite direction, so that the total path of the photon is $L - \Delta x$, and not just L. In addition, there is also a relativistic effect, the Lorentz contraction due to the velocity *v* of the box. So the path becomes:

$$L\sqrt{\left(1 - \frac{v^2}{c^2}\right)} - \Delta x$$

This leads to:

$$\Delta t = \frac{L\sqrt{\left(1 - \frac{v^2}{c^2}\right)} - \Delta x}{c}$$

$$M\frac{\Delta x}{\Delta t} = \frac{E}{c}$$

$$M\Delta x = \frac{E}{c}\,\Delta t$$

So:

$$M\Delta x = \frac{E\left(L\sqrt{\left(1 - \frac{v^2}{c^2}\right)} - \Delta x\right)}{c^2}$$

Now:

$$\frac{Mx_1 + mx_2}{M + m} = \frac{M(x_1 - \Delta x) + m\left(L\sqrt{\left(1 - \frac{v^2}{c^2}\right)} - \Delta x\right)}{M + m}$$

$$\Rightarrow -M\Delta x + m\left(L\sqrt{\left(1 - \frac{v^2}{c^2}\right)} - \Delta x\right) = 0$$

$$m\left(L\sqrt{\left(1 - \frac{v^2}{c^2}\right)} - \Delta x\right) = M\Delta x = \frac{E\left(L\sqrt{\left(1 - \frac{v^2}{c^2}\right)} - \Delta x\right)}{c^2}$$

$$\frac{E\left(L\sqrt{\left(1 - \frac{v^2}{c^2}\right)} - \Delta x\right)}{c^2} = m\left(L\sqrt{\left(1 - \frac{v^2}{c^2}\right)} - \Delta x\right)$$

$$\boldsymbol{E = mc^2}$$

Fortunately, it ends up in the same equation phew...

Appendix 9.10 Applications

Appendix 9.10.1 Nuclear Fusion and Nuclear Fission

When a proton *p* and a neutron *n* are brought together, they can fuse and form a nucleus of deuterium (also called heavy water) *d*. The masses of *p*, *n*, and *d* are:

$$m_p = 938.27231\ MeV/c^2$$

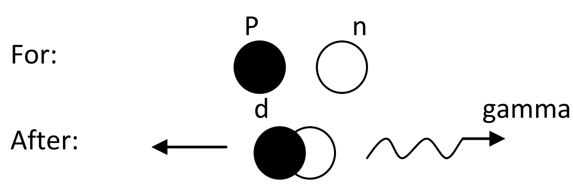
$$m_n = 939.56563\ MeV/c^2$$

$$m_d = 1875.61339\ MeV/c^2$$

The unit used, $\frac{MeV}{c^2}$, requires some explanation. From the relationship $E = mc^2$, we can see that mass can be expressed in units of energy divided by a constant c^2 (the speed of light). In 'MKSA' units, the unit of energy is the Joule, but it is also possible, and common in particle physics, to choose the electron volt (*eV)*. An electron volt is the amount of energy that a unit charge receives when it passes a potential difference of 1 Volt. The unit charge (charge of the electron) is equal to$1.6.10^{-19}$ Coulomb, so $1eV = 1.6.10^{-19}J, 1\ MeV = 10^6 eV$.

Because the mass of the deuteron (= deuterium nucleus) is smaller than the sum of the masses of its components, the proton and neutron, energy must have been released! If *p* and *n* are brought together at a negligible speed, the energy released is equal to:

$$\begin{aligned} E &= m_p c^2 + m_n c^2 - m_d c^2 \\ &= 2.22455\ MeV \end{aligned}$$



This energy is released in the form of a photon:

$$p + n \rightarrow d + \gamma$$

A photon has no mass; it is a quantum of the electromagnetic field, introduced by Einstein to explain the photoelectric effect; it is symbolized by γ . Not all of the missing mass goes into the energy of the photon. Even if *p* and *n* are at rest relative to each other before the reaction, theγ will fly away at the speed of light after the reaction. And to ensure momentum conservation, d will move in the opposite direction with the same momentum (see figure above). Due to the size of the mass of d, the energy associated with this is very small.

$$\text{Because if } pc \ll mc^2, \text{then } E = \sqrt{p^2c^2 + m^2c^4} \cong mc^2$$

The reaction described above is an example of **nuclear fusion**. In general, it appears that light nuclei can fuse into heavier nuclei while releasing energy, as in the example above. All nuclei up to and including iron can be produced via fusion while releasing energy.
The opposite effect is that heavier nuclei, such as the well-known example of uranium, are heavier than the sum of the components of the nucleus. In that case, energy is only released when the nuclei are split (**nuclear fission**).

Appendix 9.10.2 Electric Car Running on 1 gram of Hydrogen through Nuclear Fusion

Here we will look at nuclear fusion as it occurs in the sun, where four hydrogen atoms fuse to form a helium atom. In this fusion process, mass disappears and is converted into energy. We will now calculate how much energy this is and what that means in practical terms.

To determine how many kilometers you can drive on 1 gram of hydrogen through nuclear fusion, we need to go through a number of steps to calculate the energy generated by nuclear fusion and then convert it into practical driving distance.

1. **Energy yield from nuclear fusion**:

The fusion of hydrogen nuclei, as in the sun, usually occurs via the proton-proton cycle. The net reaction is:

$$4\,{}_1^1H \rightarrow {}_2^4He + 2e^+ + 2\nu_e + 2\gamma$$

In this process, in which four hydrogen atoms are converted into one helium atom, the total weight of the four hydrogen atoms is greater than the weight of the helium atom. So a small amount of mass has disappeared. Using the formula $E = mc^2$, we can calculate how much mass is converted into energy.

The energy released in this reaction is approximately 26.7 MeV (megaelectronvolts) per fusion of four hydrogen atoms into one helium atom.

1 gram of hydrogen contains approximately $6{,}022\ x\ 10^{23}$ (Avogadro's number) hydrogen atoms (1 mol). So, in 1 gram of hydrogen, we have $6{,}022\ x\ 10^{23}/4 \approx 1{,}505\ x\ 10^{23}$ fusion reactions.

Each fusion reaction yields 26.7 MeV of energy, so the total energy is:

$$E_{fusie} = 1{,}505\ x\ 10^{23} x\ 26{,}7\ \text{MeV}$$

2. **Converting MeV to Joules**:

One Joule is equal to moving a charge of 1 Coulomb in a field of 1 Volt. So

$$Joule = qV$$

The charge of an electron *e* is $1{,}60218\ x\ 10^{-19} Coulomb$.
Therefore:

$$1\,Joule = \frac{1}{1{,}60218\ x\ 10^{-19}}\ eV$$

$$=> eV = 1{,}602\ x\ 10^{-19}\ Joules$$

Therefore:

$$1MeV = 1{,}60218\ x\ 10^{-13}\ Joules$$

So, per 1 gram of hydrogen, the total energy in Joules is:

$$E_{tot} = 1{,}505\ x\ 10^{23} x\ 26{,}7\ \text{x}\ 1{,}60218\ x\ 10^{-13}\ \approx 6{,}43\ x\ 10^{11}\ Joules$$

3. **Calculation of the energy**:

$$E_{tot} \approx 6{,}43 \; x \; 10^{11} \text{ Joules per gram waterstof}$$

This is the energy released in this process, whereby a small portion of the mass is converted into energy.

For comparison, we can look at the theoretical calculation when 1 gram of matter is completely converted according to $E = mc^2$:

$$E = \frac{1}{1000} \; x \; (3x10^8)^2 \approx 9 \; x \; 10^{13} \; Joules$$

So this makes a difference of about a factor of 140 (or, in percentages, fusion accounts for 0.7% of the energy in the total conversion of 1 gram of mass).

4. **Alternative calculation**:

In nuclear fusion, 4 moles of hydrogen are converted into 1 mole of helium, releasing a quantity of energy. In this process, a small amount of mass is lost and converted into energy.

The mass of 4 moles of hydrogen H is 4x1.00784=4.03136 grams of hydrogen.

The mass of 1 mole of helium He is 4.0026 grams of helium.

So the mass difference is $0{,}02876 \text{ gram} = 2{,}876 * 10^{-5} \text{ kg.}$

$$E = mc^2 \text{ waarbij } c = 3 * 10^8 \; m/s$$

The energy released is therefore:

$$E = 2{,}876 * 10^{-5} * (3 * 10^8)^2 = 2{,}588 \; 10^{12} \;\; \text{bij } 4{,}03136 \text{ gram hydrogen}$$

That is:

$$E = 6{,}42 * 10^{11} \text{ Joule per 1 gram hydrogen}$$

1.1. **Energy consumption of an electric car**:

Electric cars consume approximately 15-20 kWh per 100 km. We assume 17 kWh per 100 km for average use.

- $1 \text{ kWh} = 3{,}6 \text{ x } 10^6 \text{ Joules.}$

$\text{Dus, } 17 \text{ kWh} = 61{,}2 \text{ x } 10^6 \text{ Joules per } 100 \text{ km.}$

1.2. **Calculation of the theoretical driving distance**:

We can calculate the distance using the theoretically available energy:

$$\text{Distance} = \frac{6{,}43 \text{ x } 10^{11}}{61{,}2 \text{ x } 10^6} \text{ x } 100 \text{ km}$$

$$\text{Distance} \approx 1{,}05 \text{ x } 10^6 \text{ km}$$

1.3. **Energy conversion efficiency**:

However, the above calculation is based on 100% efficient energy conversion. But when converting energy from nuclear fusion into usable electricity, and then into the propulsion of an electric car, we have to take into account different efficiencies:

- **Efficiency of energy conversion to electricity**: Let's assume that this efficiency is 40% (a conservative estimate, as nuclear fusion reactors are still under development).
- **Efficiency of the electric drive**: Electric cars typically have an efficiency of around 85-90%. We will use 90% for the calculation here.

The total efficiency is therefore $0{,}4\ x\ 0.9 = 0{,}36$.

1.4. **Usable energy**:

The usable energy that is ultimately available to power the car:

$$E_{usable} = 6{,}43\ x\ 10^{11}\ x\ 0{,}36 = 2{,}31\ x\ 10^{11} \text{ Joules}$$

1.5. **Calculation of the practical driving distance**:

We can calculate the distance using the practical available energy:

$$\text{Afstand} = \frac{2{,}31 \text{ x } 10^{11}}{61{,}2 \text{ x } 10^6} \text{ x } 100 \text{ km}$$

$$\text{Afstand} \approx 3{,}77\text{x}10^5 \text{ km}$$

Therefore, an electric car powered by the energy from the nuclear fusion of 1 gram of hydrogen can theoretically drive approximately 377 thousand **(377,000) kilometers**. The average distance generally traveled per year is approximately 15,000 km, which means that **1 gram of hydrogen can power a car for twenty-five (25) years**.

Appendix 9.11 Relativistic electromagnetism

(Calculations based on Richard Feynman https://www.feynmanlectures.caltech.edu/II_13.html)

Appendix 9.11.1 Introduction

The word electromagnetism assumes that there is an electric field and a magnetic field, and therefore suggests that there are sources for both fields. However, we know that electric charge is the source of the electric field, and so far no magnetic sources have been found for the magnetic field. It seems that a magnetic field is always caused by an electric field that varies over time. Even on a microscopic scale, the quantum scale, magnetic fields are caused by the electric spins of electrons or atoms. The electric field has electrons (-1e) and protons (+1e) as its sources.

Perhaps we can go so far as to say that the magnetic field model is only a very useful mathematical tool for describing the electromagnetic phenomenon; but the only thing that exists is the electric field and the variation of the electric field based on accumulations of electrons and protons.

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Appendix 9.11.2 Calculations

If we take the example of a current carrier, we can normally use Maxwell's equations to calculate the electric and magnetic fields. An alternative approach is to base the calculation entirely on the electric field and skip the magnetic part.

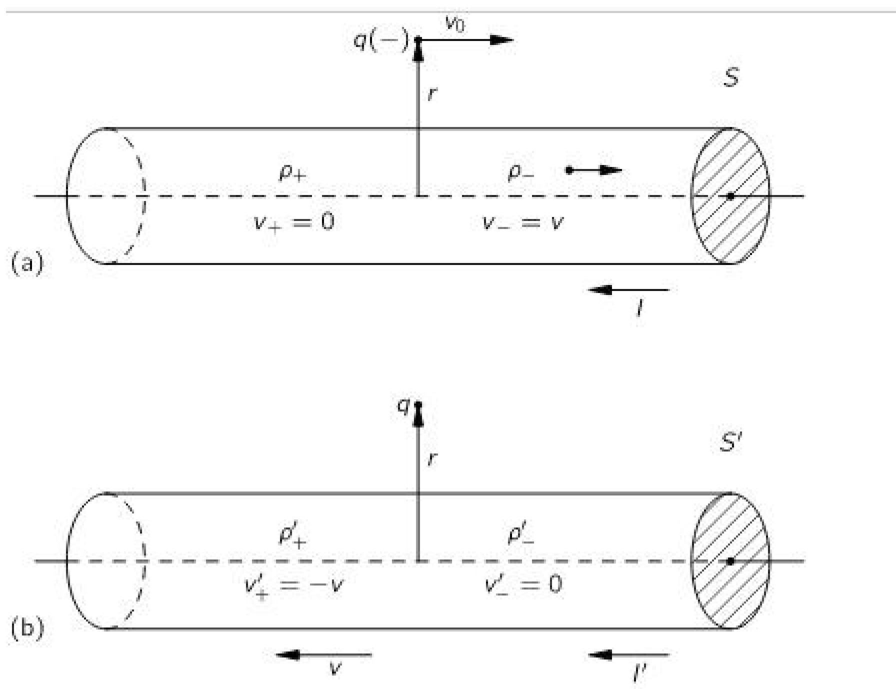


Fig. 1. The interaction of a current carrier and a particle with charge q, seen in two frames. In frame S (part a), the wire is at rest; in frame S' (part b), the charge is at rest.

We will now derive some formulas for later use.

The current density is the average flow velocity of the charges. Suppose there is a distribution of charges with an average velocity v. The distribution passes over a surface element ΔS , the charge Δq that passes through the surface element in a time Δt is equal to the charge:

$$\Delta q = \rho \boldsymbol{v} \cdot \boldsymbol{n} \Delta S \Delta t \tag{1}$$

Here, ρ is the charge density: the charge per unit volume.

Here, $\boldsymbol{v}\Delta t \cdot \Delta \boldsymbol{S}$ can be considered as a volume. So, the charge is the charge density multiplied by the volume.
The charge per unit time is then $\rho \boldsymbol{v} \cdot \boldsymbol{n} \Delta S$, , which gives:

$$\boldsymbol{j} = \rho \boldsymbol{v} \tag{2}$$

The total current through the surface S is:

$$\boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{S} \tag{3}$$

We now consider a current carrier that is at rest and electrons, negatively charged particles, that move to the right at a speed v. The protons, positively charged particles, remain at rest in the wire. A test particle, with a negative charge q, moves to the right at the same speed as the electrons. We observe the whole system at rest relative to the wire. The total wire distributes all charges in such a way that it is neutral.

Let us consider the external force on the wire that can be caused by the electric and magnetic fields:

$$F = q(E + v \times B)$$
$$B = \mu_0 H$$

The magnetic field around the wire is:

$$H = \frac{i}{2\pi r}$$

Consider the force on the test particle where the electric field is zero because the total charge in the wire is neutral:

$$F = q(v \times B) = qvB \sin \varphi$$

Since v is perpendicular to B, then $\sin \varphi = 1$

$$F = qvB = qv\mu_0 H = \frac{qv\mu_0 i}{2\pi r}$$

The charge density ρ is defined as the total charge in a volume divided by the size of the volume V :

$$\rho = \frac{q}{V}$$

If A is the cross-sectional area of the wire and L is the arbitrarily chosen length of the volume, at rest, along the wire, then:

$$q = \rho A L$$

When the wire is at rest:

$$\rho_+ + \rho_- = 0$$

If we now consider the situation from the perspective of the test particle, the test particle is at rest and the wire moves to the left at a speed v.
The volume is determined by A and its length L. The length between a moving volume and a volume at rest is:

$$L_{moving} = L_{rest}\sqrt{1-\frac{v^2}{c^2}}$$

Since the speed of the electrons is the same as the chosen speed of the test particle, the electrons are now also at rest. This means that:

$$L_{rest} = \frac{L_{moving}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Because the wire now has a velocity of *v* to the left, the positive particles also move at *v* to the left, and the length *L* of the volume changes by a factor of:

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

When the wire was at rest, the external electric field outside the wire was:

$$\rho_+ + \rho_- = \rho_+ - \rho_+ = 0$$

Because the moving length is smaller than the rest length, the moving volume is also smaller. So then the density of the charged particles is greater. So, if we consider the charge density when the test particle is at rest, we must multiply the moving density ρ_- by

$$\sqrt{1-\frac{v^2}{c^2}}$$

So, now the electric field outside the wire is determined by the total charge density:

$$\rho_{nett} = \frac{\rho_+}{\sqrt{1-\frac{v^2}{c^2}}} + \rho_-\sqrt{1-\frac{v^2}{c^2}} = \frac{\rho_+}{\sqrt{1-\frac{v^2}{c^2}}} - \rho_+\sqrt{1-\frac{v^2}{c^2}} = \rho_+\frac{1-1+\frac{v^2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\rho_{nett} = \rho_+\frac{\frac{v^2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

The volume of a length *L* of the wire gives a charge of:

$$q = \rho_+\frac{\frac{v^2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}AL$$

So, the electric field outside the wire is not zero and is perpendicular to the wire. If we consider a tube around the wire of length *L* and distance from the axis of the wire of *r*, the volume is:

$$E = \rho_+\frac{\frac{v^2}{c^2}}{2\pi\varepsilon_0 rL\sqrt{1-\frac{v^2}{c^2}}}AL = \rho_+\frac{\frac{v^2}{c^2}}{2\pi\varepsilon_0 r\sqrt{1-\frac{v^2}{c^2}}}A$$

So:

$$F^{'} = qE = q\rho_+ \frac{\frac{v^2}{c^2}}{2\pi\varepsilon_0 r\sqrt{1-\frac{v^2}{c^2}}}A \tag{4}$$

For *v* much smaller than *c*:

$$F^{'} = qE = q\rho_+ \frac{1}{2\pi\varepsilon_0 r}\frac{v^2}{c^2}A$$

For the magnetic field in the rest state, it was:

$$F = qvB = qv\mu_0 H = \frac{qv\mu_0 i}{2\pi r} \tag{5}$$

This gives, when *J* is the current density through the wire and $J = \rho v$:

$$F = \frac{qv\mu_0 i}{2\pi r} = \frac{qv\mu_0 JA}{2\pi r}$$

$$c^2 = \frac{1}{\varepsilon_0\mu_0} => \mu_0 = \frac{1}{\varepsilon_0 c^2}$$

$$F = \frac{qv\mu_0 JA}{2\pi r} = \frac{qv\rho vA}{2\pi r\varepsilon_0 c^2} = \frac{qv\rho vA}{2\pi r\varepsilon_0 c^2} = q\rho\frac{1}{2\pi\varepsilon_0 r}\frac{v^2}{c^2}A \tag{6}$$

So from equations (4) and (6) it follows that:

$$F^{'} = \frac{F}{\sqrt{1-\frac{v^2}{c^2}}}$$

The forces act in the transverse y-direction, so the momentum in the y -direction and the $y^{'}$ -direction must be the same because the transverse velocity is zero.

Now we compare the momentum in the y and $y^{'}$ directions:

$$\Delta p_y = F\Delta t$$

And

$$\Delta p'_y = F'\Delta t'$$

As we know, time seems to pass more slowly for a moving particle than in the particle's rest frame, so:

$$\Delta t = \frac{\Delta t'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Delta p_y = \Delta p'_y = F\Delta t = F\frac{\Delta t'}{\sqrt{1-\frac{v^2}{c^2}}} = F'\Delta t'$$

$$=> F^{'} = \frac{F}{\sqrt{1-\frac{v^2}{c^2}}} \tag{7}$$

So now it has been shown, with equations (5), (6), and (7), that:

$$F = qvB = \sqrt{1 - \frac{v^2}{c^2}} F' = \sqrt{1 - \frac{v^2}{c^2}} qE$$

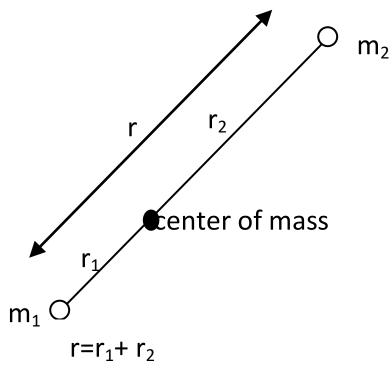
Appendix 9.11.3 Conclusion

We have found that we get the same physical result regardless of whether we analyze the motion of a particle moving along a wire in a coordinate system that is at rest relative to the wire, or in a system that is at rest relative to the particle. In the first instance, the force was purely "magnetic," while in the second instance it was purely "electric." This also shows that magnetism is in fact a relativistic effect.

Appendix 10 Specific Angular Momentum

In this document, and especially where we use the Schwarzschild equation, the term angular momentum is used. This is denoted by the form $L = mr^2 \frac{d\phi}{dt}$; because $\left(L = mvr = mrv = mr\frac{rd\phi}{dt} = mr^2 \frac{d\phi}{dt} \right)$.

However, it is not the actual angular momentum, but an approximation. An explanation follows below.
In the Schwarzschild formula, there is a relationship between a particle and a large, massive body. The chosen frame of reference is the center of the large, massive body. It is therefore a kind of two-body problem. Let us now consider the angular momentum for a two-body problem.



The two bodies rotate around each other and the center of gravity is called the barycenter. The condition for circular bodies is that:

$$\frac{m_1 v_1^2}{r_1} = \frac{m_2 v_2^2}{r_2} \tag{1}$$

To ensure symmetry of forces, the masses must remain on opposite sides of the barycenter. The periods of the orbits must therefore be equal:

$$T = \frac{2\pi r_1}{v_1} = \frac{2\pi r_2}{v_2} \quad \Longrightarrow \frac{\boldsymbol{v_1}}{\boldsymbol{v_2}} = \frac{\boldsymbol{r_1}}{\boldsymbol{r_2}} \tag{2}$$

$$v_1 = \frac{r_1}{r_2} v_2 = \frac{r_1}{r_2}(v - v_1) \tag{3}$$

$$v_1 \left(1 + \frac{r_1}{r_2}\right) = \frac{r_1}{r_2} v \quad \Longrightarrow \boldsymbol{v_1} = \frac{\boldsymbol{r_1}}{\boldsymbol{r}} \boldsymbol{v} \tag{4}$$

And similarly:

$$v_2 = \frac{r_2}{r_1} v_1 \quad \Rightarrow \boldsymbol{v_2} = \frac{\boldsymbol{r_2}}{\boldsymbol{r}} \boldsymbol{v} \tag{5}$$

m_2 ten opzichte van m_1 's velocity is:

$$v = v_1 + v_2 \tag{6}$$

Substitute (3) into (1):

$$\frac{m_1 v_1^2}{r_1} = \frac{m_1 v_2^2}{r_1}\left(\frac{r_1}{r_2}\right)^2 = \frac{m_2 v_2^2}{r_2} \ \Longrightarrow \frac{m_1}{r_1}\left(\frac{r_1}{r_2}\right)^2 = \frac{m_2}{r_2}$$

$$\Longrightarrow \boldsymbol{m_1 r_1 = m_2 r_2} \tag{7}$$

$$m_2 r_2 = m_1(r - r_2) = m_1 r - m_1 r_2$$

$$r_2(m_1 + m_2) = m_1 r \quad \Rightarrow \boldsymbol{r_2} = \frac{\boldsymbol{m_1}}{\boldsymbol{m_1} + \boldsymbol{m_2}} \boldsymbol{r} \tag{8}$$

Let's calculate the angular momentum of m_2 relative to m_1 .

$$L_2 = m_2 v_2 r_2 = m_2 \frac{r_2}{r} v \frac{m_1}{m_1 + m_2} r = m_2 \frac{m_1}{m_1 + m_2} r_2 v = m_2 \quad \left(\frac{m_1}{m_1 + m_2}\right)^2 vr \tag{9}$$

$$L_2 = \frac{1}{m_2} \left(\frac{m_1 m_2}{m_1 + m_2}\right)^2 \omega r^2 \tag{10}$$

$$L_1 = \frac{1}{m_1} \left(\frac{m_1 m_2}{m_1 + m_2}\right)^2 \omega r^2 \tag{11}$$

The total angular momentum of the two bodies:

$$L = L_2 + L_1 = \left(\frac{1}{m_2} + \frac{1}{m_1}\right) \left(\frac{m_1 m_2}{m_1 + m_2}\right)^2 \omega r^2 = \frac{m_1 + m_2}{m_1 m_2} \left(\frac{m_1 m_2}{m_1 + m_2}\right)^2 \omega r^2 = \frac{m_1 m_2}{m_1 + m_2} \omega r^2$$

To bring it in line with the Schwarzschild equation:

$$L = \frac{m_1 m_2}{m_1 + m_2} r^2 \frac{d\Phi}{d\tau} \tag{12}$$

We call *m* the reduced mass.

$$m = \frac{m_2 m_1}{m_1 + m_2} \tag{13}$$

The specific angular momentum *h* is:

$$h = \frac{L}{m} = r^2 \frac{d\Phi}{d\tau} \tag{14}$$

In the case where m_1 represents a large mass *M* and m_2 represents the mass of a particle, then:

$$\frac{m_2 M}{M + m_2} \Rightarrow m_2 \tag{15}$$

So, if $M \gg m_2$, then the mass in the angular momentum equation is determined by the mass of the particle alone.

Appendix 11 Considerations about Rotation

Appendix 11.1 Introduction

Below, we will explain centrifugal and centripetal force, first based on Newton, and later we will extend this to general relativity. Centrifugal force is the force that acts outward from the center of rotation. Centripetal force is directed toward the center.

Appendix 11.2 Momentum

According to Newton, a moving particle with mass *m* and velocity *v* has momentum *mv;* if no forces act on the particle, it will move uniformly in a straight line at velocity *v*. Relative to a point at distance *r,* the particle has angular momentum $m\vec{v} \times \vec{r}$.

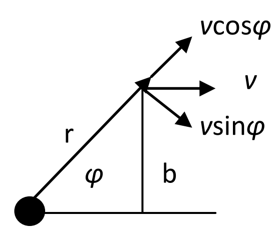


Fig. 1

In the image above, the angular momentum is $L = m\,\mathrm{vsin}\,\varphi\,.\,r = mvr\sin\varphi$ or $L = mvb$.

Appendix 11.3 Circle

As mentioned earlier, the particle will move uniformly in a straight line, so if the particle's trajectory is a circle, a force is required.

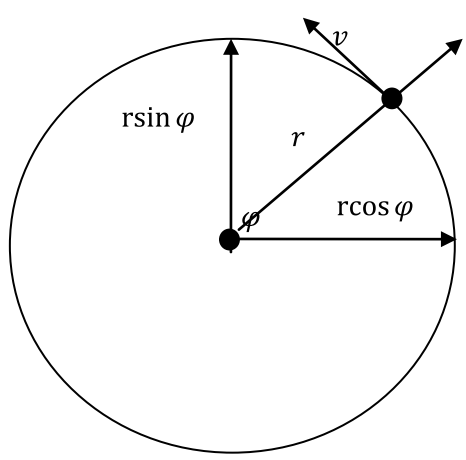


Fig. 2

We start with a constant radius *r* and split this into the *x* and y components. From there, we calculate the circular velocity and acceleration:

$$x = r\cos\varphi = r\cos\omega t$$

$$y = r\sin\varphi = r\sin\omega t$$

$$v_x = \frac{dx}{dt} = -\omega r \sin \omega t$$

$$v_y = \frac{dy}{dt} = \omega r \cos \omega t$$

$$a_x = \frac{d^2x}{dt^2} = -\omega^2 r \cos \omega t$$

$$a_y = \frac{d^2y}{dt^2} = -\omega^2 r \sin \omega t$$

The total force is:

$$F = m\sqrt{{a_x}^2 + {a_y}^2} = -m\omega^2 r$$

So, the particle wants to move along a straight line, but due to its rotation, it feels a perpendicular outward force $F = m\omega^2 r$. As shown above, this force must be compensated by a centripetal reaction force $F = -m\omega^2 r$, i.e., in the direction of the center, to keep the particle on its circular path.

Appendix 11.4 Rotation of a Sphere

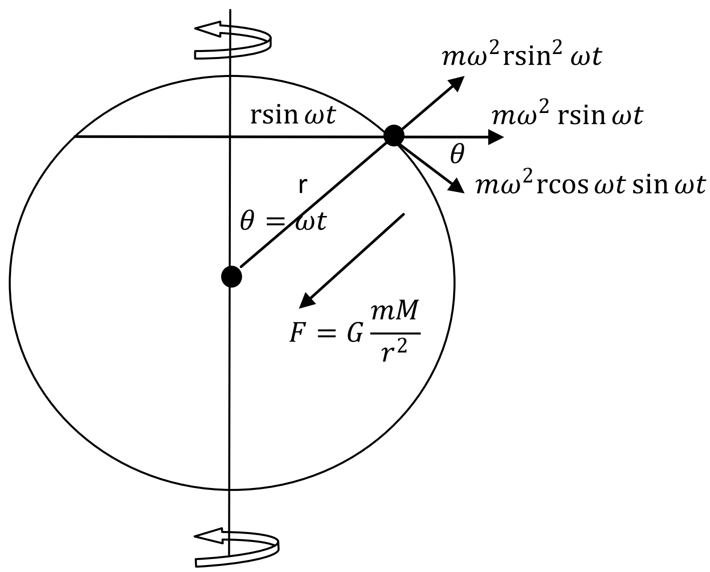


Fig. 3

In principle, we consider the sphere to consist of many particles, but we will now initially consider the behavior of a single particle somewhere on the sphere.

The particle rotates around the vertical axis and experiences a horizontal centrifugal force of:

$$m\omega^2\, \mathrm{rsin}\, \omega t$$

This results in a centrifugal force in the radial direction of the sphere of:

$$m\omega^2 \mathrm{rsin}^2\, \omega t$$

Together with the centripetal force, this results in the following force:

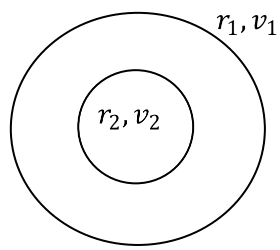
$$G\frac{mM}{r^2} - m\omega^2 \mathrm{rsin}^2\, \omega t = m\left(\frac{GM}{r^2} - \omega^2 \mathrm{rsin}^2\, \omega t\right)$$

In addition, there is a tangential force towards the equator of:

$$m\omega^2 \mathrm{r}\cos\omega t \sin\omega t$$

Particles will therefore feel a force towards the equator, which will cause the sphere to deform into an ellipsoid. This means that the distance from the center to the particle is shortest at the poles and longest at the equator; as a result, the gravitational force differs per location. Gravity also depends on the enclosed mass; since the distance from the poles to the center is the smallest, the enclosed mass is smallest there. So, gravity at the poles increases due to the smaller distance, but decreases due to the enclosed mass. The deformation of the sphere will result in an ellipsoid where equilibrium is achieved.
(see also: http://farside.ph.utexas.edu/teaching/336k/Newton/node109.html).

Appendix 11.5 Relationship between Angular Momentum and Energy



Difference in kinetic energy of the two circles:

$$\Delta K = \frac{1}{2}m{v_1}^2 - \frac{1}{2}m{v_2}^2 \tag{1}$$

The angular momentum is constant, so:

$$mv_1r_1 = mv_2r_2$$

$$v_2 = \frac{v_1r_1}{r_2} \tag{2}$$

Now (2) in (1):

$$\Delta K = \frac{1}{2}m{v_1}^2 - \frac{1}{2}m\left(\frac{v_1r_1}{r_2}\right)^2 = \frac{1}{2}m{v_1}^2\left(1 - \frac{{r_1}^2}{{r_2}^2}\right) \tag{3}$$

This energy difference ΔK must be supplied by the centripetal force:

$$F = -\frac{mv^2}{r}$$

Energy is:

$$\int_{r_2}^{r_1} F dr = -\int_{r_2}^{r_1} \frac{mv^2}{r} dr$$

The angular momentum is constant, so:

$$mvr = Const$$

$$v = \frac{Const}{mr}$$

$$\int_{r_2}^{r_1} F dr = -\int_{r_2}^{r_1} \frac{m}{r}\frac{Const^2}{m^2r^2} dr = -\int_{r_2}^{r_1} \frac{Const^2}{mr^3} dr = \frac{Const^2}{2mr^2}\Big|_{r_2}^{r_1} = \frac{Const^2}{2m}\left(\frac{1}{{r_1}^2} - \frac{1}{{r_2}^2}\right)$$

$$= \frac{m^2 {v_1}^2 {r_1}^2}{2m}\left(\frac{1}{{r_1}^2} - \frac{1}{{r_2}^2}\right) = \frac{1}{2} m {v_1}^2 \left(1 - \frac{{r_1}^2}{{r_2}^2}\right) \qquad (4)$$

We see that formulas (3) and (4) are equal, so:

$$\Delta K = \int_{r_2}^{r_1} F dr = \frac{1}{2} m {v_1}^2 \left(1 - \frac{{r_1}^2}{{r_2}^2}\right)$$

Appendix 12 Derivation of the Euler-Lagrange equation

We start with a function f_1 that depends on three variables: t, x_1 en $\frac{dx_1}{dt}$:

$$f_1 = f\left(t, x_1(t), \frac{dx_1(x,t)}{dt}\right) \quad of \quad f_1 = f(t, x_1, \dot{x}_1) \qquad (1)$$

Here,x_1 is a function of *t,* so $\frac{dx_1(t)}{dt}$ is not zero. In principle, *t* is the only variable that determines the function f_1 . So f_1 is a function of a function.

Now we consider the function f_1 between the points t_1 en t_2 . We now integrate f_1 between these points:

$$I_1 = \int_{t_1}^{t_2} f_1 \, dt \qquad (2)$$

$$I_1 = \int_{t_1}^{t_2} f\left(t, x_1(t), \frac{dx_1(t)}{dt}\right) dt$$

To find the extreme value (maximum, saddle point, or minimum) of I_1 , the following applies:

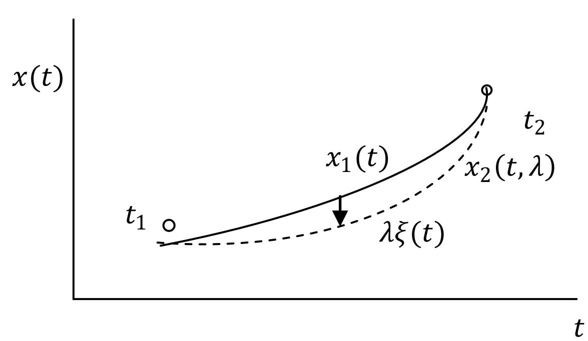
$$\delta I_1 = 0 \qquad (3)$$

To prove that I_1 is an extreme value, we consider a curve $x_2(t)$ that is slightly shifted so that I_2 is not an extreme value:

$$x_2(t, \lambda) = x_1(t) + \lambda \xi(t) \qquad (4)$$

Here, λ is a parameter that is independent of *t*. Because we are considering a curve that runs from t_1 till t_2 ,$x_2(t)$ differs from$x_1(t)$ between these points, but at the points $t_1$en t_2 , $x_1(t) = x_2(t)$ applies. So:

$$\xi(t_1) = 0 \text{ and } \xi(t_2) = 0 \qquad (5)$$



Now, the integral I_2 for the adjacent curve is:

$$I_2 = \int_{t_1}^{t_2} f_2 \, dt \tag{6}$$

$$I_1 = \int_{t_1}^{t_2} f\left(t, x_2(t,\lambda), \frac{dx_2(t,\lambda)}{dt}\right) dt$$

Substituting (6) into equation (4), we obtain:

$$I_2 = \int_{t_1}^{t_2} f\left(t, x_2(t,\lambda), \frac{dx_2(t,\lambda)}{dt}\right) dt \tag{7}$$

$$= \int_{t_1}^{t_2} f\left(t, x_1(t) + \lambda\xi(t), \frac{d(x_1(t) + \lambda\xi(t))}{dt}\right) dt$$

$$= \int_{t_1}^{t_2} f\left(t, x_1(t) + \lambda\xi(t), \frac{dx_1(t)}{dt} + \lambda\frac{d\xi(t)}{dt}\right) dt$$

Since I_1 is an extreme value, I_2 is also extreme for $\lambda = 0$:

$$\frac{lim}{\lambda \to 0} I_2 = \text{minimum, zadelpunt of maximum} \tag{8}$$

The extreme value can be found by taking the derivative and setting it equal to zero:

$$\frac{lim}{\lambda \to 0} \frac{dI_2}{d\lambda} = 0 \tag{9}$$

In combination with equation (6):

$$\frac{lim}{\lambda \to 0} \frac{d}{d\lambda}\left(\int_{t_1}^{t_2} f_2 \, dt\right) = 0 \tag{10}$$

$$\frac{lim}{\lambda \to 0} \int_{t_1}^{t_2} \frac{d}{d\lambda}(f_2 dt) = 0$$

This is a product of two functions, so we apply the rule of partial differentiation:

$$\frac{lim}{\lambda \to 0} \int_{t_1}^{t_2} \left(\frac{df_2}{d\lambda} dt + f_2 \frac{d(dt)}{d\lambda}\right) = 0 \tag{11}$$

Because$t$ en λ are mutually independent, the derivative of t naar λ , or vice versa, is equal to zero:

$$\frac{lim}{\lambda \to 0} \int_{t_1}^{t_2} \left(\frac{df_2}{d\lambda} dt + f_2 . 0\right) = 0 \tag{12}$$

$$\frac{lim}{\lambda \to 0} \int_{t_1}^{t_2} \frac{df_2}{d\lambda} dt = 0$$

Next, we apply the rule of partial differentiation:

$$\frac{lim}{\lambda\to 0}\int_{t_1}^{t_2}\left(\frac{\partial f_2}{\partial t}\frac{dt}{d\lambda}+\frac{\partial f_2}{\partial x_2}\frac{dx_2}{d\lambda}+\frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)}\frac{d\left(\frac{dx_2}{dt}\right)}{d\lambda}\right)dt\ =0 \tag{13}$$

Due to the independence of t en λ , the first term is zero:

$$\frac{lim}{\lambda\to 0}\int_{t_1}^{t_2}\left(\frac{\partial f_2}{\partial t}.0+\frac{\partial f_2}{\partial x_2}\frac{dx_2}{d\lambda}+\frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)}\frac{d\left(\frac{dx_2}{dt}\right)}{d\lambda}\right)dt\ =0 \tag{14}$$

$$\frac{lim}{\lambda\to 0}\int_{t_1}^{t_2}\left(\frac{\partial f_2}{\partial x_2}\frac{dx_2}{d\lambda}+\frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)}\frac{d\left(\frac{dx_2}{dt}\right)}{d\lambda}\right)dt\ =0$$

Because:

$$\frac{d\left(\frac{dx_2}{dt}\right)}{d\lambda}=\frac{d^2x_2}{dtd\lambda}=\frac{d\left(\frac{dx_2}{d\lambda}\right)}{dt} \tag{15}$$

Equation (14), together with (15), leads to:

$$\frac{lim}{\lambda\to 0}\int_{t_1}^{t_2}\left(\frac{\partial f_2}{\partial x_2}\frac{dx_2}{d\lambda}+\frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)}\frac{d\left(\frac{dx_2}{d\lambda}\right)}{dt}\right)dt\ =0 \tag{16}$$

$$\frac{lim}{\lambda\to 0}\left(\int_{t_1}^{t_2}\frac{\partial f_2}{\partial x_2}\frac{dx_2}{d\lambda}dt+\int_{t_1}^{t_2}\frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)}\frac{d\left(\frac{dx_2}{d\lambda}\right)}{dt}dt\right)=0$$

Now we integrate the right-hand side of this equation:

$$\int_{t_1}^{t_2}\frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)}\frac{d\left(\frac{dx_2}{d\lambda}\right)}{dt}dt=\int_{t_1}^{t_2}\frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)}d\left(\frac{dx_2}{d\lambda}\right) \tag{17}$$

$$=\left[\frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)}\frac{dx_2}{d\lambda}\right]_{t_1}^{t_2}-\int_{t_1}^{t_2}\frac{dx_2}{d\lambda}\frac{\partial}{\partial t}\left(\frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)}\right)dt$$

The derivative of f x_2 naar λ is found by differentiating equation (4):

$$\frac{dx_2(t,\lambda)}{d\lambda}=\frac{d(x_1(t)+\lambda\xi(t))}{d\lambda}=0+\xi(t)=\xi(t) \tag{18}$$

Because the function $\xi(t)$ is zero at the boundaries of the integral (see equation (5)), the left side of the right-hand term in equation (17) disappears:

$$\int_{t_1}^{t_2}\frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)}\frac{d\left(\frac{dx_2}{d\lambda}\right)}{dt}dt=\left[\frac{\partial f_2}{\partial\left(\frac{dx_2}{dx}\right)}\frac{dx_2}{d\lambda}\right]_{t_1}^{t_2}-\int_{t_1}^{t_2}\frac{dx_2}{dt}\frac{\partial}{\partial t}\left(\frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)}\right)dt$$

$$= -\int_{t_1}^{t_2} \frac{dx_2}{dt}\frac{\partial}{\partial t}\left(\frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)}\right)dt \tag{19}$$

This result combined with equation (16) leads to:

$$\frac{lim}{\lambda \to 0}\left(\int_{t_1}^{t_2} \frac{\partial f_2}{\partial x_2}\frac{dx_2}{d\lambda}dt + \int_{t_1}^{t_2} \frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)}\frac{d\left(\frac{dx_2}{d\lambda}\right)}{dt}dt\right) = 0 \tag{20}$$

$$\frac{lim}{\lambda \to 0}\left(\int_{t_1}^{t_2} \frac{\partial f_2}{\partial x_2}\frac{dx_2}{d\lambda}dt - \int_{t_1}^{t_2} \frac{dx_2}{d\lambda}\frac{d}{dt}\left(\frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)}\right)dt\right) = 0$$

$$\frac{lim}{\lambda \to 0}\left(\int_{t_1}^{t_2} \left(\frac{\partial f_2}{\partial x_2}\frac{dx_2}{d\lambda} - \frac{dx_2}{d\lambda}\frac{d}{dt}\left(\frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)}\right)\right)dt\right) = 0$$

$$\frac{lim}{\lambda \to 0}\left(\int_{t_1}^{t_2} \left(\frac{\partial f_2}{\partial x_2} - \frac{d}{dt}\left(\frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)}\right)\right)\frac{dx_2}{d\lambda}dt\right) = 0$$

To make this integral zero, we set:

$$\frac{lim}{\lambda \to 0}\left(\frac{\partial f_2}{\partial x_2} - \frac{d}{dt}\left(\frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)}\right)\right) = 0$$

$$\frac{\partial f_1}{\partial x_1} - \frac{d}{dt}\left(\frac{\partial f_1}{\partial\left(\frac{dx_1}{dt}\right)}\right) = 0 \tag{21}$$

Now λ has completely disappeared and we have obtained a general expression for the condition that a function must satisfy in order for the integral I to have an extreme value.
We started with equation (1) for our derivation, but we could make this starting point even more general by taking a function such as:

$$f_1 = f\left(t, x_1(t), \frac{dx_1(t)}{dt}, x_2(t), \frac{dx_2(t)}{dt}, \dots \dots \dots,, x_n(t), \frac{dx_n(t)}{dt}\right) \qquad 22$$

This would have led to a more general form of equation (21):

$$\frac{\partial f}{\partial x_n} - \frac{d}{dt}\left(\frac{\partial f}{\partial\left(\frac{dx_n}{dt}\right)}\right) = 0 \tag{23}$$

Or in another notation:

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_n}\right) = \frac{\partial f}{\partial x_n} \tag{24}$$

Equation (24) is the **Euler-Lagrange equation**. It specifies the condition that a function must satisfy in order for the integral I to be an extreme value.

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EINSTEIN'S GENERAL THEORY OF RELATIVITY

A Tensor-Analytical Approach

with

Experimental Verification

by

Albert Prins

END

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